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Mathematical Models of Some Parasitic Diseases Involving an Intermediate Host

Ingemar Nåsell and Warren M. Hirsch

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Abstract

Mathematical models are established for two classes of parasitic diseases. In each case the disease is caused by a parasite which reaches sexual maturity inside its definitive host and which requires one intermediate host for the completion of its life cycle. In one case the mature parasite is hermaphroditic, while in the other case it has a monogamous heterosexual mating pattern.

From each model we derive an autonomous non-linear system of differential equations, whose solutions describe the course of the disease in a community in which it is endemic. We establish existence in the large, uniqueness, and boundedness of the solutions of these systems, and we study the critical points and their stability properties. From these results we deduce the influence of various biological and environmental parameters on the steady-state prevalence of the disease, on the proportion of infected intermediate hosts, and on a measure of public health. This makes it possible to compare the effectiveness of various procedures aimed at control or eradication of the disease and reveals the import of the parasite's mode of mating for the efficacy of such procedures.

Chapter 1

Biological Background

1.1 Parasitic Diseases

In the Animal Kingdom there exist many species that are parasitic. A parasite spends part or all of its life in close association with a different species of animal or plant. The parasite lives on or in its host, and feeds itself at the expense of its host, but does not usually kill it or destroy it rapidly. In many cases, however, the parasite inflicts injury on its host in addition to what is caused by its dependence on the host for nourishment. In fact, a large number of diseases are caused by parasites, and many of these have medical or veterinary importance since human beings or domesticated animals are affected. Among the parasitic animals there are some that have complicated life cycles that require two or more different species of hosts for their completion. When this is the case, the parasite reaches sexual maturity in one of the hosts. In this host, sexual mating takes place and eggs are produced. The host species is referred to as the definitive host for the parasite. In the other host species, reproduction is asexual, and the corresponding hosts are referred to as intermediate hosts for the parasite.

In this report we restrict ourselves to a study of parasitic diseases that involve one definitive host and one intermediate host. The pattern for sexual mating in the definitive host can take different forms, depending on the biology of the sexually mature parasites. We consider two different cases. In one, the parasites are hermaphroditic and every mature parasite can produce eggs. In the other case, male and female organs appear in different individuals, and monogamous heterosexual mating is required in order for the female parasite to produce eggs.

We give below descriptions of life cycles for two specific parasite species. In one of these (Fasciola hepatica), the mature parasites are hermaphroditic, and in the other one (Schistosoma mansoni), the mating in the definitive host is monogamous heterosexual. The diseases caused by these two organisms are illustrative of the two types of diseases for which models are established in Chapters 6 and 8.

A detailed discussion of the biology of animal parasites is found in Cheng [3].

1.2 The Life Cycle of Schistosoma mansoni

The parasite whose life cycle is described in this section belongs to the phylum Platyhelminthes, the class

Trematoda, the order Digenea, the family Schistosomatidae, and the genus Schistosoma. The members of this phylum are called flatworms. The flatworms of the class Trematoda are all parasitic. They are commonly referred to as flukes. The digenetic trematodes have some of the most complex life cycles of any animal. Each of them utilizes one definitive host and at least one intermediate host. In the genus Schistosoma are found several species whose adult forms live in blood vessels of human beings. Three wide-spread such species are S. haematobium, S. japonicum, and S. mansoni. The life cycles of these three species, although not identical, are very similar. In this section we describe the life cycle of Schistosoma mansoni.

Human beings serve as definitive hosts for S. mansoni, while certain species of snails serve as intermediate hosts. The adult forms of the parasite are found in the veins of the intestines of the definitive host. The male and female forms are permanently paired. The male is 6-10 mm long, and the female measures 7-14 mm in length. The male is broader than the female. He carries her in a groove (the gynecophoric canal) on his body. The rate of egg output of a paired female is difficult to determine; Hairston [9] gives the estimate 140 per day. The longevity of adult parasites is estimated to average about three years; cases have, however, been reported where worms have achieved the age of 20 years.

Some of the eggs deposited by the female will penetrate the wall of the intestine and later leave the body of the host with the feces. Other eggs that do not achieve this success become trapped in various tissues, such as the liver, and cause the disease symptoms. Some of the eggs that leave the body of the host come in contact with fresh water before they die. When this happens, the eggs hatch, and from each egg a free-swimming larva, called a miracidium, emerges.

The miracidium has a short life-span (24 hours or less). If during its lifetime a miracidium reaches a snail of suitable species, it penetrates the snail and initiates an asexual phase of reproduction inside the body of the snail. This phase results in the production of a second type of free-swimming organism, called a cercaria. Average daily outputs of cercariae per infected snail are reported in the range 130-4,600; see Chu-Dawood [4]. All the cercariae that are offspring of one miracidium have the same sex.

Like the miracidia, cercariae have a short life expectancy. If during its lifetime a cercaria comes sufficiently close to a human being, it will penetrate its skin and migrate to the liver. Here it grows to adult size before migrating to the intestinal blood vessel, where sexual maturation and mating occur.

The life-cycle of S. mansoni is thus seen to involve two reproductive stages where the rates of reproduction are very high, interlaced with two short free-swimming stages with very small probabilities of survival. The relative magnitudes of these reproduction rates and survival probabilities in a given community will clearly influence the size of the existing parasite population. In the mathematical model of Chapter 8 we shall see that death rates of mature parasites and recovery rates of intermediate hosts also play an important role.

Detailed discussions of a number of aspects of schistosomiasis (the disease caused by schistosome parasites) are given in Wolstenholme - O'Connor [17].

The symptoms of schistosomiasis include late afternoon fever, nonproductive cough, watery skin eruptions, swelling and pain of the liver, diarrhea, hemorrhages, and occasionally tumors in brain or other organs. Schistosomiasis is not always a direct cause of death. It leads, however, to general disability and weakness. A person suffering from schistosomiasis may therefore easier become afflicted with other diseases. It is estimated that more than 200,000,000 persons suffer from schistosomiasis. Because it is so widespread and has such crippling effects, schistosomiasis is a serious world health problem. This fact serves as one motivation for the work on mathematical models presented here.

1.3 The Life Cycle of Fasciola hepatica

The parasite Fasciola hepatica is a digenetic trematode belonging to the family Fasciolidae. It is commonly referred to as a liver fluke. In its definitive host, mainly sheep, it causes liver damage.

Mature forms are hermaphroditic. They reside in the bile ducts of their definitive host. Some eggs leave the body of the host with the feces. If they reach fresh water, miracidia develop and escape. The life length of a miracidium is about eight hours. Its main goal in life is achieved if in this time it is able to find and penetrate a snail of a specific species. Inside the snail, an asexual reproductive stage gives rise to cercariae which leave the snail and become free-swimming. When a free-swimming cercaria reaches aquatic vegetation it changes form into an immobile cyst called metacercaria that attaches itself to the vegetation. The life cycle is completed when the definitive host swallows a metacercaria. Inside the host, the metacercaria excysts and migrates to the bile duct where it matures and starts egg production.

We use the term "hermaphroditic helminthiasis" to refer to a parasitic disease for which the mature parasite is hermaphroditic and requires one intermediate host for the completion of its life cycle. The disease caused by Fasciola hepatica is an example of hermaphroditic helminthiasis.

The purpose of the present report is to establish and study mathematical models of two classes of disease, namely hermaphroditic helminthiasis and schistosomiasis. The diseases in these two classes are similar; they differ only in the mating of mature parasites. Thus we shall be able to study the epidemiological differences between these diseases caused by the different mating patterns for mature parasites.

Assumptions and objectives for the establishment of the mathematical models are discussed in Chapter 2.

Chapter 2

Modeling Objectives

The models we develop all refer to an ecological complex consisting of a community of potential definitive hosts for the parasites, the environment in which they live, and a collection of potential intermediate hosts for the parasites residing in this environment. In order to be able to establish mathematical models we introduce parameters that describe the biology and environment of hosts and parasites.

The assumptions made in the establishment of the models are of necessity of a simplifying nature. Our aim is to ignore details that are judged to be of small consequence, but to incorporate those features that are judged to be significant. We assume a homogeneous mix of definitive and intermediate hosts throughout the community for which our models are established. This means that individual differences in exposure to the environment between different definitive or intermediate hosts are ignored. It means also that living miracidia and cercariae are uniformly distributed throughout the environment. We furthermore assume that both definitive and intermediate hosts are continuously exposed to the environment. Another assumption made is that the latent periods can be ignored in both definitive and intermediate hosts. This means that we disregard the time period

between the moment when a cercaria penetrates a definitive host and the moment when the cercaria has grown to a sexually mature parasite. It means also that we assume an intermediate host to start producing cercariae as soon as it has been penetrated by a miracidium. Other assumptions are that the populations of both definitive and intermediate hosts are constant in time. We assume furthermore that the recovery rate of infected intermediate hosts is independent of the length of the infectious period, and that the rate of output of cercariae from an infected intermediate host is constant throughout the period when it remains infected. For hermaphroditic helminthiasis we assume that the rate of egg output per mature parasite is constant. In the case of schistosomiasis, the corresponding assumption is that the rate of egg output from a paired mature female parasite is constant as long as it remains paired and alive. The death rates of mature parasites are assumed also to be independent of age and of sex.

We emphasize one implication of the above assumptions. Multiple infections of definitive hosts are clearly accounted for, since the total egg output in the community is related to the number of egg-laying parasites and not to the number of definitive hosts. On the other hand, even though multiple infections of intermediate hosts can occur, the output of cercariae is related to the number

of infected intermediate hosts, and a host is classified as infective as soon as it has been penetrated by one miracidium. Thus, the model does not assume that the cercarial output of an intermediate host will increase as the result of an infection following the first one. A justification for this assumption in the case of schistosomiasis is given by Macdonald [14].

We note also that the model does not assume that immunity toward parasites develops in the definitive hosts.

In the mathematical investigation of the properties of our models we put special emphasis on properties that have meaningful epidemiological interpretations. The totality of simplifying assumptions we have made may lead one to question the quantitative predictions of the models. However, the qualitative features of the results are in themselves of considerable interest. We point especially to the threshold effects and control results whose epidemiological interpretations are discussed in Chapters 7 and 9.

In establishing models of the two classes of disease of interest here, we proceed as follows: Chapter 3 gives a stochastic model of the egg-producing parasites alive in the body of a definitive host, while Chapter 4 gives a stochastic model of the infected intermediate hosts. These separate host models are of interest in themselves. Furthermore, they serve an important role in both the

establishment and the discussion of the stochastic-deterministic models for the two classes of disease treated in Chapters 6 and 8, respectively.

The subject matter of the report belongs to the area of mathematical epidemiology. The term "epidemiology" refers to that field of medical science which studies factors that influence the spread of a disease in a community. Broadly speaking, communicable diseases are of two kinds, epidemic and endemic, although the demarcation line between the two is not sharp. In describing these two kinds of disease we use the term "prevalence" to denote the proportion of persons in a community that are afflicted by the disease at a given instance of time. An epidemic disease is characterized by long troughs in prevalence, followed by high peaks signifying brief outbreaks of the disease affecting a large number of persons in a short time interval. An endemic disease, on the other hand, is characterized by a rather constant prevalence, which can be high or low, over long time periods. Even though this description is somewhat imprecise, it is useful to indicate the kinds of problems and questions that are of concern to an epidemiologist. Thus, for epidemic diseases one is interested in the factors that influence the size of an epidemic, i.e. the total number of persons that become infected in one outbreak of the disease, and, more generally, the factors that influence the entire functional relationship between prevalence and

time during one outbreak of the disease. On the other hand, for endemic diseases one is interested in the factors that influence the endemic level of prevalence of the disease. From the standpoint of mathematical modeling, epidemic diseases give rise to the study of transient phenomena, while endemic diseases lead us to study steady-state levels.

The diseases of concern in this report are endemic. In the models that we establish we investigate the relation between the various biological-environmental parameters that we introduce and the resulting levels of prevalence. This problem, and some related ones, are studied in Chapters 7 and 9.

Reviews of the theory of mathematical epidemiology have been given by Bailey [2] and Dietz [7]. The book by Bailey covers developments up to 1957, while the paper by Dietz reviews contributions during the decade 1957-1967. Neither of these references deals directly with the diseases that we discuss here. Works devoted specifically to mathematical modeling of schistosomiasis have been published by Goffman-Warren [8], Hairston [9], [10], Leyton [13], Macdonald [14], and Tallis-Leyton [16].

Chapter 3

Model for Definitive Host

3.1 Model Formulation

In this chapter we establish for each definitive host models for the number of mature parasites and the number of mature paired parasites residing inside its body. These models are based on simpler models for the number of female and male parasites, respectively. Details in their formulation are given below.

The following notation will be used in this chapter: At each time $t \geq 0$ the random variables $\xi_F(t)$ and $\xi_M(t)$ denote, respectively, the number of female and male parasites alive in the body of the host. Based on these random variables we define four more as follows:

$$\xi(t) = \xi_F(t) + \xi_M(t) \tag{3.1.1}$$

$$\xi_P(t) = 2(\xi_F(t) \wedge \xi_M(t))^{*} \tag{3.1.2}$$

$$\xi_{FS}(t) = (\xi_F(t) - \tfrac{1}{2}\,\xi_P(t)) \tag{3.1.3}$$

$$\xi_{MS}(t) = (\xi_M(t) - \tfrac{1}{2}\,\xi_P(t)) \tag{3.1.4}$$

Here, $\xi(t)$ denotes the total number of parasites at time t, $\xi_P(t)$ is the number of paired parasites at time t, $\xi_{FS}(t)$ is the number of single females at time t, and

---

* We use the notation $a \wedge b = \min(a,b)$ and $a \vee b = \max(a,b)$.

$\xi_{MS}(t)$ is the number of single males at time t. These definitions are based on the assumption that no time is required for two mature parasites of opposite sex to find each other and form a pair. The following two relations among these random variables follow immediately from the definitions:

$$\xi(t) = \xi_{FS}(t) + \xi_{MS}(t) + \xi_{P}(t) \tag{3.1.5}$$

$$\xi_{FS}(t) \wedge \xi_{MS}(t) = 0. \tag{3.1.6}$$

The remainder of this section is devoted to the establishment of models for $\xi_F(t)$ and $\xi_M(t)$. The succeeding sections of the chapter study the distributions of the above random variables.

As model for $\xi_F(t)$ we propose a continuous time, discrete state, Markov process, whose transition probabilities satisfy the conditions given below. We refer to this process as a non-homogeneous immigration-death process.

The transition probability functions are denoted by

$$P_{ij}(t,u) = P\{\xi_F(u) = j \mid \xi_F(t) = i\},$$

$$i,j = 0,1,2,\ldots; \quad 0 \leq t < u.$$

We postulate the following forms of these functions for $u - t = \Delta t$ as $\Delta t \to 0$:

$$P_{f,f+1}(t,t+\Delta t) = \tfrac{1}{2}\nu_1 Y(t)\Delta t + o(\Delta t),$$

$$P_{f,f}(t,t+\Delta t) = 1 - (\tfrac{1}{2}\nu_1 Y(t) + f\mu_1)\Delta t + o(\Delta t),$$

$$P_{f,f-1}(t,t+\Delta t) = f\mu_1 \Delta t + o(\Delta t);$$

here, $\nu_1 > 0$ and $\mu_1 > 0$ are constants, while $Y(t) \geq 0$ is a bounded continuous function defined for $t \geq 0$. We refer to μ_1 as the death rate per parasite, and to $\frac{1}{2}\nu_1 Y(t)$ as the immigration rate of the process. We assume that $\xi_F(0) = f_o$, where f_o is a non-negative integer.

The probability distribution of $\xi_F(t)$ is denoted by $P_f(t) = P\{\xi_F(t)=f\} = P_{f_o,f}(0,t)$, $f=0,1,2,\ldots$ Using the above death rate and immigration rate, the following differential-difference equations (Kolmogoroff forward equations) are established for the function $P_f(t)$:

$$P_f'(t) = \tfrac{1}{2}\nu_1 Y(t) P_{f-1}(t)$$

$$- (\tfrac{1}{2}\nu_1 Y(t) + f\mu_1) P_f(t)$$

$$+ (f+1)\mu_1 P_{f+1}(t), \qquad f=0,1,2,\ldots, \; t \geq 0; \tag{3.1.7}$$

here, we define $P_{-1}(t) \equiv 0$. At time $t = 0$ we have

$$P_f(0) = \begin{cases} 1, & f = f_o, \\ 0, & f \neq f_o. \end{cases}$$

The model for the number of male parasites $\xi_M(t)$ is a non-homogeneous immigration-death process with the same immigration rate and death rate as above. The initial condition is $\xi_M(0) = m_o$, where m_o is a non-negative integer. We assume that the random variables $\xi_F(t)$ and $\xi_M(t)$ are independent. The probability distribution of $\xi_M(t)$ is denoted by $Q_m(t) = P\{\xi_M(t)=m\}$, $m=0,1,2,\ldots$ The functions $Q_m(t)$ satisfy the following differential-difference equations:

$$Q_m'(t) = \tfrac{1}{2}\nu_1 Y(t) Q_{m-1}(t) - (\tfrac{1}{2}\nu_1 Y(t) + m\mu_1) Q_m(t) + (m+1)\mu_1 Q_{m+1}(t), \quad m=0,1,2,\ldots,\ t \geq 0; \tag{3.1.8}$$

we define $Q_{-1}(t) \equiv 0$. At $t = 0$ we have

$$Q_m(0) = \begin{cases} 1, & m = m_o, \\ 0, & m \neq m_o. \end{cases}$$

The model for the total number of parasites $\xi(t)$ is a non-homogeneous immigration-death process whose immigration rate is twice the immigration rate for each of the two processes above, while the death rate per parasite equals μ_1. The initial condition is $\xi(0) = n_o = f_o + m_o$. The probability distribution of $\xi(t)$ is denoted by $R_n(t) = P\{\xi(t)=n\}$, $n=0,1,2,\ldots$ The functions $R_n(t)$ satisfy the following differential-difference equations:

$$R_n'(t) = \nu_1 Y(t) R_{n-1}(t)$$

$$- (\nu_1 Y(t) + n\mu_1) R_n(t)$$

$$+ (n+1)\mu_1 R_{n+1}(t), \qquad n=0,1,2,\ldots;\ t \geq 0; \qquad (3.1.9)$$

we define $R_{-1}(t) \equiv 0$. At $t = 0$ we have

$$R_n(0) = \begin{cases} 1, & n = n_o, \\ 0, & n \neq n_o. \end{cases}$$

The system of differential equations (3.1.7) with its initial data has a unique solution given by the sequence of functions $P_f(t)$ with $f = 0,1,2,\ldots$ and $t \geq 0$, provided we impose the requirements that the component functions $P_f(t)$ be non-negative and that $\sum_{f=0}^{\infty} P_f(t) \leq 1$. These results follow by a straight-forward extension of the arguments given by Reuter-Ledermann [15]. The same argument shows existence and uniqueness of the solutions of the initial-value problems for which the differential equations are given in (3.1.8) and (3.1.9), respectively.

3.2 The Distributions of Females and Males

We introduce the following notation:

$$\alpha(t) = e^{-\mu_1 t}, \qquad (3.2.1)$$

$$\beta(t) = \nu_1 e^{-\mu_1 t} \int_0^t Y(s) e^{\mu_1 s}\, ds\ . \qquad (3.2.2)$$

The following lemma gives the distribution and expectation of $\xi_F(t)$, the number of females at time t.

Lemma 3.2.1

$$P\{\xi_F(t) = f\}$$

$$= e^{-\frac{1}{2}\beta(t)} \sum_{i=0}^{f_o} \binom{f_o}{i} (\alpha(t))^i (1-\alpha(t))^{f_o-i} \frac{(\frac{1}{2}\beta(t))^{f-i}}{(f-i)!} \quad *$$

and

$$E\xi_F(t) = \frac{1}{2}\beta(t) + f_o\,\alpha(t).$$

Proof: We introduce the probability generating function

$$G(t,z) = \sum_{f=0}^{\infty} P_f(t) z^f,$$

where the functions $P_f(t)$ satisfy the differential-difference equations of the preceding section. We note that the derived series $\sum_{f=0}^{\infty} P_f'(t)z^f$ and $\sum_{f=1}^{\infty} f\,P_f(t)z^{f-1}$ converge uniformly for $|z| \leq 1 - \varepsilon$, $\varepsilon > 0$. Hence $G(t,z)$ satisfies the partial differential equation

$$\frac{\partial G(t,z)}{\partial t} + \mu_1(z-1)\frac{\partial G(t,z)}{\partial z} = \frac{1}{2}\nu_1 Y(t)(z-1)G(t,z),$$

with initial data specified along the line $t = 0$ by $G(0,z) = z^{f_o}$. The characteristics are solutions of the equation

$$\frac{dz}{dt} = \mu_1(z-1).$$

---

* We adopt the conventions $\binom{N}{n} = 0$ for $n < 0$ or $n > N$, and $\frac{1}{n!} = 0$ for $n < 0$.

Any solution of this equation can be written

$$z(t) = 1 + K e^{\mu_1 t},$$

where K is a constant.

To solve for G(t,z) we consider the ordinary differential equation

$$G'(t) = \frac{1}{2}\nu_1 KY(t)e^{\mu_1 t} G(t)$$

with initial condition

$$G(0) = (1+K)^{f_o}.$$

This is the condition to be satisfied by G along a characteristic determined by K. The solution is

$$G(t) = (1+K)^{f_o} \exp\left\{\frac{1}{2}\nu_1 K \int_0^t Y(s)e^{\mu_1 s} ds\right\}.$$

The solution of the initial value problem posed above for the function G(t,z) can therefore be written

$$G(t,z) = G_N(t,z)\, G_I(t,z),$$

where

$$G_N(t,z) = [1+\alpha(t)(z-1)]^{f_o}$$

and

$$G_I(t,z) = \exp(\tfrac{1}{2}\beta(t)(z-1)).$$

Here we have made use of the functions $\alpha(t)$ and $\beta(t)$, defined in (3.2.1) and (3.2.2), respectively. We recognize that $G_N(t,z)$ is the probability generating function of a random variable $\xi_N(t)$, whose distribution is binomial with parameters f_o and $\alpha(t)$. Furthermore, $G_I(t,z)$ is the probability generating function of a random variable $\xi_I(t)$, whose distribution is Poisson with parameter $\frac{1}{2}\beta(t)$. It follows that $\xi_F(t)$ can be expressed as the sum of the independent random variables $\xi_N(t)$ and $\xi_I(t)$:

$$\xi_F(t) = \xi_N(t) + \xi_I(t).$$

By convolving the distributions of $\xi_N(t)$ and $\xi_I(t)$, we find the distribution of $\xi_F(t)$ to be the one asserted in the lemma. By adding the expectations of $\xi_N(t)$ and $\xi_I(t)$, we find the expectation of $\xi_F(t)$. QED

We note that the random variables $\xi_N(t)$ and $\xi_I(t)$ can be interpreted as the number of survivors of natives and immigrants, respectively.

The preceding lemma allows us to write down expressions for the distributions and the expected values of the random variables $\xi_M(t)$ and $\xi(t)$. In particular, we find

$$P\{\xi(t) = 0\} = e^{-\beta(t)} (1-\alpha(t))^{f_o+m_o} \tag{3.2.3}$$

and

$$E\xi(t) = \beta(t) + n_o\alpha(t). \tag{3.2.4}$$

3.3 The Distributions of Single Females and Males

In this section the modified Bessel functions $I_n(x)$, with n an integer, play an important role. We quote the definition and some basic properties from Abramowitz-Stegun [1]:

$$I_n(x) = \sum_{k=0}^{\infty} \frac{(\frac{x}{2})^{2k+n}}{k!(k+n)!}; \tag{3.3.1}$$

$$I_n(x) = I_{-n}(x); \tag{3.3.2}$$

$$xI_{n-1}(x) - xI_{n+1}(x) = 2nI_n(x); \tag{3.3.3}$$

$$I_{n-1}(x) + I_{n+1}(x) = 2I_n'(x); \tag{3.3.4}$$

$$I_0(x) + 2\sum_{n=1}^{\infty} I_n(x) = e^x; \tag{3.3.5}$$

$$I_n(x) \sim \frac{e^x}{\sqrt{2\pi x}} \quad (x\to\infty); \tag{3.3.6}$$

$$I_n(x) = \frac{1}{\pi}\int_0^{\pi} e^{x\cos\theta}\cos(n\theta)d\theta. \tag{3.3.7}$$

From (3.3.1) we get for $n \geq 0$

$$I_{n+m}(x) = \sum_{k=0}^{\infty} \frac{(\frac{x}{2})^{2k-n+m}}{(k-n)!(k+m)!}\ . \tag{3.3.8}$$

From Cochran [5] it follows that

$$I_n(x) < I_m(x), \quad n > m \geq 0,\ x > 0. \tag{3.3.9}$$

The following lemma will be applied later in this section:

Lemma 3.3.1 Let ℓ be an integer. Then

$$\sum_{k=1}^{\infty} kI_{k+\ell}(x) = \frac{x}{2}\,(I_0(x)+I_1(x)) - \frac{\ell}{2}\,e^x + \frac{|\ell|}{2}\,I_0(x) + \sum_{k=1}^{|\ell|-1} (|\ell|-k)I_k(x).^*$$

Proof: Let $\ell=0$. Then, using recursion formula (3.3.3) and relation (3.3.5), we get

$$\sum_{k=1}^{\infty} kI_k(x) = \frac{x}{2}\sum_{k=0}^{\infty} I_k(x) - \frac{x}{2}\sum_{k=2}^{\infty} I_k(x)$$

$$= \frac{x}{2}\,(I_0(x) + I_1(x)). \qquad (3.3.10)$$

For any integer ℓ we have

$$\sum_{k=1}^{\infty} kI_{k+\ell}(x) = \sum_{k=\ell+1}^{\infty} kI_k(x) - \ell \sum_{k=\ell+1}^{\infty} I_k(x).$$

Now let $\ell > 0$. Then, by applying (3.3.5) and (3.3.10) we get

$$\sum_{k=1}^{\infty} kI_{k+\ell}(x) = \frac{x}{2}\,(I_0(x)+I_1(x)) - \sum_{k=1}^{\ell} kI_k(x) - \frac{\ell}{2}\,e^x + \frac{\ell}{2}\,I_0(x) + \ell \sum_{k=1}^{\ell} I_k(x).$$

Finally, let $\ell < 0$. Then, applying (3.3.2), (3.3.5), and (3.3.10), we get

$$\sum_{k=1}^{\infty} kI_{k+\ell}(x) = \sum_{k=\ell+1}^{-1} kI_k(x) + \frac{x}{2}\,(I_0(x)+I_1(x))$$

(cont.)

* We adopt the convention $\sum_{k=m}^{n} \alpha_k = 0$ for $n < m$.

$$- \ell \sum_{k=\ell+1}^{-1} I_k(x) - \ell I_0(x) - \frac{\ell}{2}(e^x - I_0(x))$$

$$= \frac{x}{2}(I_0(x)+I_1(x)) - \frac{\ell}{2} e^x - \frac{\ell}{2} I_0(x)$$

$$- \sum_{k=1}^{-\ell-1} kI_k(x) - \ell \sum_{k=1}^{-\ell-1} I_k(x). \qquad \text{QED}$$

The next lemma concerns the distribution of the random variable $\xi_{MS}(t)$.

Lemma 3.3.2 Let $k \geq 1$. Then

$$P\{\xi_{MS}(t) = k\}$$

$$= \sum_{i=0}^{f_o} \sum_{j=0}^{m_o} \binom{f_o}{i}\binom{m_o}{j} (\alpha(t))^{i+j} (1-\alpha(t))^{f_o+m_o-i-j}$$

$$\cdot e^{-\beta(t)} I_{k+i-j}(\beta(t)).$$

Proof: For $k \geq 1$ we have

$$\{\xi_{MS}(t)=k\} = \bigcup_{f=0}^{\infty} [\{\xi_M(t)=f+k\} \cap \{\xi_F(t)=f\}].$$

Using independence of $\xi_M(t)$ and $\xi_F(t)$ and Lemma 3.2.1 we get (omitting the argument t):

$$P\{\xi_{MS}=k\}$$

$$= \sum_{f=0}^{\infty} e^{-\beta} \sum_{j=0}^{m_o} \binom{m_o}{j} \alpha^j (1-\alpha)^{m_o-j} \sum_{i=0}^{f_o} \binom{f_o}{i} \alpha^i (1-\alpha)^{f_o-i}$$

$$\cdot \frac{(\frac{1}{2}\beta)^{2f+k-i-j}}{(f-i)!(f+k-j)!}.$$

The conclusion of the lemma now follows via an application of (3.3.8). QED

We now proceed to evaluate the expected number of single male parasites $E\xi_{MS}(t)$. In the evaluation we make use of the six functions g_1, g_2, g_3, g_4, g_5, and G, whose definitions are given below in (3.3.11)-(3.3.16). Each of these functions has four arguments α, β, f_o and m_o. We assume that $0 \leq \alpha \leq 1$, $0 \leq \beta$, and that f_o, m_o are non-negative integers. Wherever the undetermined expression 0^0 appears in the definitions, we define it to have the value 1. The definitions are

$$g_1(\alpha,\beta;f_o,m_o)$$

$$= \sum_{k=1}^{m_o-1} e^{-\beta} I_k(\beta) \sum_{i=0}^{f_o} \sum_{j=k+i+1}^{m_o} (j-k-i) \binom{f_o}{i} \binom{m_o}{j} \alpha^{i+j} (1-\alpha)^{f_o+m_o-i-j}; \tag{3.3.11}$$

$$g_2(\alpha,\beta;f_o,m_o)$$

$$= e^{-\beta} I_0(\beta) \sum_{i=0}^{f_o} \sum_{j=i+1}^{m_o} (j-i) \binom{f_o}{i} \binom{m_o}{j} \alpha^{i+j} (1-\alpha)^{f_o+m_o-i-j}$$

$$+ 2\, g_1(\alpha,\beta;f_o,m_o); \tag{3.3.12}$$

$$g_3(\alpha,\beta;f_o,m_o)$$

$$= e^{-\beta} I_0(\beta) \sum_{i=0}^{f_o} \sum_{j=0}^{m_o} |j-i| \binom{f_o}{i} \binom{m_o}{j} \alpha^{i+j} (1-\alpha)^{f_o+m_o-i-j}; \tag{3.3.13}$$

$$g_4(\alpha,\beta;f_o,m_o)$$

$$= e^{-\beta}I_0(\beta) \sum_{i=1}^{f_o} \sum_{j=1}^{m_o} (i \wedge j) \binom{f_o}{i}\binom{m_o}{j} \alpha^{i+j}(1-\alpha)^{f_o+m_o-i-j}; \qquad (3.3.14)$$

$$g_5(\alpha,\beta;f_o,m_o)$$

$$= \sum_{i=0}^{f_o} \sum_{j=0}^{m_o} \binom{f_o}{i}\binom{m_o}{j} \alpha^{i+j}(1-\alpha)^{f_o+m_o-i-j} \sum_{k=1}^{|i-j|-1} (|i-j|-k)e^{-\beta}I_k(\beta); \qquad (3.3.15)$$

and

$$G(\alpha,\beta;f_o,m_o)$$

$$= (f_o+m_o)\alpha - g_2(\alpha,\beta;f_o,m_o) - g_2(\alpha,\beta;m_o,f_o). \qquad (3.3.16)$$

Some interrelations between these functions are given in the next three lemmas. The first of these lemmas contains the functions g_3 and g_4.

Lemma 3.3.3

$$g_3(\alpha,\beta;f_o,m_o)$$

$$= (f_o+m_o)e^{-\beta}I_0(\beta)\alpha - 2g_4(\alpha,\beta;f_o,m_o).$$

Proof: Insert the identity $|j-i| = i + j - 2(i \wedge j)$ into the expression (3.3.13) for the function g_3. The conclusion of the lemma then follows. QED

An alternative form for the function G is given by

Lemma 3.3.4

$$G(\alpha,\beta;f_o,m_o)$$

$$= (f_o+m_o)\alpha - g_3(\alpha,\beta;f_o,m_o) - 2g_1(\alpha,\beta;f_o,m_o) - 2g_1(\alpha,\beta;m_o,f_o).$$

Proof: Using the definitions (3.3.16) and (3.3.12) we get

$$G(\alpha,\beta;f_o,m_o) = (f_o+m_o)\,\alpha - 2g_1(\alpha,\beta;f_o,m_o)$$

$$-2g_1(\alpha,\beta;m_o,f_o)$$

$$-e^{-\beta}I_0(\beta) \sum_{i=0}^{f_o} \sum_{j=i+1}^{m_o} (j-i) \binom{f_o}{i}\binom{m_o}{j} \alpha^{i+j}(1-\alpha)^{f_o+m_o-i-j}$$

$$-e^{-\beta}I_0(\beta) \sum_{j=0}^{m_o} \sum_{i=j+1}^{f_o} (i-j) \binom{f_o}{i}\binom{m_o}{j} \alpha^{i+j}(1-\alpha)^{f_o+m_o-i-j}.$$

But, for any numbers γ_{ij},

$$\sum_{i=0}^{f_o} \sum_{j=i+1}^{m_o} (j-i)\,\gamma_{ij} + \sum_{j=0}^{m_o} \sum_{i=j+1}^{f_o} (i-j)\,\gamma_{ij}$$

$$= \sum_{i=0}^{f_o} \sum_{j=0}^{m_o} |j-i|\,\gamma_{ij}.$$

Hence the lemma follows. QED

We note that Lemmas 3.3.3 and 3.3.4 can be combined to give another alternative form for the function G.

The functions g_1 and g_5 are closely related, as shown by the following lemma.

Lemma 3.3.5

$$g_5(\alpha,\beta;f_o,m_o) = g_1(\alpha,\beta;f_o,m_o) + g_1(\alpha,\beta;m_o,f_o).$$

Proof:

The function g_5 is symmetric in its arguments f_o and m_o. We assume therefore, without loss of generality, that $f_o \geq m_o$. Now, for arbitrary numbers γ_{ij},

$$\sum_{i=0}^{f_o}\sum_{j=0}^{m_o}\gamma_{i,j} = \sum_{n=0}^{m_o}\sum_{i=0}^{m_o-n}\gamma_{i,i+n}$$

$$+ \sum_{n=1}^{f_o-m_o}\sum_{j=0}^{m_o}\gamma_{j+n,j} + \sum_{n=f_o-m_o+1}^{f_o}\sum_{j=0}^{f_o-n}\gamma_{j+n,j}.$$

Hence

$$g_5(\alpha,\beta;f_o,m_o)$$

$$= \sum_{n=2}^{m_o}\sum_{i=0}^{m_o}\binom{f_o}{i}\binom{m_o}{i+n}\alpha^{2i+n}(1-\alpha)^{f_o+m_o-2i-n}\sum_{k=1}^{n-1}(n-k)e^{-\beta}I_k(\beta)$$

$$+ \sum_{n=2}^{f_o-m_o}\sum_{j=0}^{f_o}\binom{f_o}{j+n}\binom{m_o}{j}\alpha^{2j+n}(1-\alpha)^{f_o+m_o-2j-n}\sum_{k=1}^{n-1}(n-k)e^{-\beta}I_k(\beta)$$

$$+ \sum_{n=f_o-m_o+1}^{f_o}\sum_{j=0}^{f_o}\binom{f_o}{j+n}\binom{m_o}{j}\alpha^{2j+n}(1-\alpha)^{f_o+m_o-2j-n}\sum_{k=1}^{n-1}(n-k)e^{-\beta}I_k(\beta).$$

The last two sums here can be combined into one, with summation over n from 2 to f_o. Next we apply the identity

$$\sum_{n=2}^{N}\sum_{k=1}^{n-1}\gamma_{kn} = \sum_{k=1}^{N-1}\sum_{n=k+1}^{N}\gamma_{kn}$$

to get

$$g_5(\alpha,\beta;f_o,m_o)$$

$$= \sum_{k=1}^{m_o-1} e^{-\beta} I_k(\beta) \sum_{n=k+1}^{m_o} \sum_{i=0}^{m_o} (n-k) \binom{f_o}{i}\binom{m_o}{i+n} \alpha^{2i+n}(1-\alpha)^{f_o+m_o-2i-n}$$

$$+ \sum_{k=1}^{f_o-1} e^{-\beta} I_k(\beta) \sum_{n=k+1}^{f_o} \sum_{j=0}^{f_o} (n-k) \binom{f_o}{j+n}\binom{m_o}{j} \alpha^{2j+n}(1-\alpha)^{f_o+m_o-2j-n}.$$

In the first sum the summation variable n is replaced by $j = i + n$, and in the second sum it is replaced by $i = j + n$. The stated result then follows. QED

We now proceed with the development of an expression for the expected value of the random variable $\xi_{MS}(t)$. The result is given the status of a theorem.

Theorem 3.3.1

$$E\xi_{MS}(t) = m_o\alpha(t)$$

$$+ \frac{1}{2}\beta(t)e^{-\beta(t)}[I_0(\beta(t))+I_1(\beta(t))]$$

$$- \frac{1}{2} G(\alpha(t), \beta(t); f_o, m_o).$$

Proof: For brevity, we omit the argument t. From Lemmas 3.3.2 and 3.3.1 we get

$$E\xi_{MS} = E_1 + E_2 + E_3 + E_4, \text{ where}$$

$$E_1 = \sum_{i=0}^{f_o} \sum_{j=0}^{m_o} \binom{f_o}{i}\binom{m_o}{j} \alpha^{i+j}(1-\alpha)^{f_o+m_o-i-j} e^{-\beta} \frac{\beta}{2} (I_0(\beta)+I_1(\beta)),$$

$$E_2 = \sum_{i=0}^{f_o} \sum_{j=0}^{m_o} \binom{f_o}{i}\binom{m_o}{j} \alpha^{i+j}(1-\alpha)^{f_o+m_o-i-j} \frac{1}{2} (j-i),$$

$$E_3 = \sum_{i=0}^{f_o} \sum_{j=0}^{m_o} \binom{f_o}{i}\binom{m_o}{j} \alpha^{i+j}(1-\alpha)^{f_o+m_o-i-j} \frac{1}{2} |i-j| e^{-\beta} I_0(\beta),$$

and

$$E_4 = \sum_{i=0}^{f_o} \sum_{j=0}^{m_o} \binom{f_o}{i}\binom{m_o}{j} \alpha^{i+j}(1-\alpha)^{f_o+m_o-i-j} \sum_{k=1}^{|i-j|-1} (|i-j|-k)e^{-\beta} I_k(\beta).$$

Now,

$$E_1 = \frac{1}{2} \beta e^{-\beta}(I_0(\beta)+I_1(\beta)),$$

$$E_2 = \frac{1}{2} (m_o-f_o)\alpha,$$

$$E_3 = \frac{1}{2} g_3(\alpha,\beta;f_o,m_o),$$

and

$$E_4 = g_5(\alpha,\beta;f_o,m_o).$$

By Lemma 3.3.5 we get

$$\begin{aligned} E\xi_{MS} &= \frac{1}{2} \beta e^{-\beta}(I_0(\beta)+I_1(\beta)) \\ &\quad + \frac{1}{2} (m_o-f_o)\alpha \\ &\quad + \frac{1}{2} g_3(\alpha,\beta;f_o,m_o) \\ &\quad + g_1(\alpha,\beta;f_o,m_o) \\ &\quad + g_1(\alpha,\beta;m_o,f_o). \end{aligned}$$

Lemma 3.3.4 now gives

$$E\xi_{MS} = \frac{1}{2}\beta\, e^{-\beta}(I_0(\beta)+I_1(\beta))$$

$$+ \frac{1}{2}(m_o-f_o)\,\alpha + \frac{1}{2}(f_o+m_o)\alpha$$

$$- \frac{1}{2} G(\alpha,\beta;f_o,m_o). \qquad \text{QED}$$

Noting that G is symmetric in f_o, m_o, we conclude that

$$E\xi_{FS}(t) = E\xi_{MS}(t) + (f_o-m_o)\,\alpha(t).$$

3.4 The Distribution of Paired Parasites

The complete distribution of the random variable $\xi_P(t)$ is not required in the sequel and will therefore not be given here. We shall, however, make use of the probability that $\xi_P(t)$ is strictly positive. Now

$$P\{\xi_P(t)=0\} = P[\{\xi_M(t)=0\}\cup\{\xi_F(t)=0\}].$$

By Lemma 3.2.1 we have

$$P\{\xi_F(t)=0\} = e^{-\frac{1}{2}\beta(t)(1-\alpha(t))f_o}.$$

Using the corresponding expression for $P\{\xi_M(t)=0\}$, we get

$$P\{\xi_P(t) > 0\}$$

$$= \left(1-e^{-\frac{1}{2}\beta(t)(1-\alpha(t))f_o}\right)\left(1-e^{-\frac{1}{2}\beta(t)(1-\alpha(t))m_o}\right). \qquad (3.4.1)$$

We next define a function ψ as follows:

$$\psi(x) = x(1-e^{-x}I_0(x)-e^{-x}I_1(x)). \tag{3.4.2}$$

Some of the properties of this function are summarized in the following lemma.

Lemma 3.4.1

(i) $\psi(x) > 0$, $x > 0$;

(ii) $\psi'(x) > 0$, $x > 0$;

(iii) $\psi''(x) > 0$, $x > 0$;

(iv) $x\psi'(x) - \psi(x) > 0$, $x > 0$;

(v) $\psi(x) \sim \frac{1}{2} x^2 \quad (x\to 0)$;

(vi) $\psi(x) \sim x \quad (x\to\infty)$.

Proof:

Using (3.3.4), (3.3.2), and (3.3.3), we get

$$\psi'(x) = 1 - e^{-x}I_0(x) \tag{3.4.3}$$

and

$$\psi''(x) = e^{-x}(I_0(x)-I_1(x)). \tag{3.4.4}$$

For $x > 0$ we have $I_n(x) > 0$ and hence, using (3.3.5), $\psi(x) > 0$ and $\psi'(x) > 0$. By applying (3.3.9) to (3.4.4) we get $\psi''(x) > 0$ for $x > 0$. From (3.4.2) and (3.4.3) we get $x\psi'(x) - \psi(x) = xe^{-x}I_1(x) > 0$ for $x > 0$. From (3.3.1) we get

$$I_n(x) \sim \frac{(\frac{x}{2})^n}{n!} \quad (x\to 0) \tag{3.4.5}$$

and hence

$$\psi(x) \sim \frac{1}{2} x^2 \qquad (x \to 0).$$

Finally, (3.3.6) gives

$$\psi(x) \sim x \qquad (x \to \infty).$$ QED

The expected value of the random variable $\xi_P(t)$ is evaluated in the following theorem.

Theorem 3.4.1

$$E\xi_P(t) = \psi(\beta(t)) + G(\alpha(t), \beta(t); f_o, m_o).$$

Proof: From relations (3.1.1) and (3.1.5) we get

$$E\xi_P(t) = E\xi_F(t) + E\xi_M(t)$$

$$-E\xi_{FS}(t) - E\xi_{MS}(t).$$

From Lemma 3.2.1 and Theorem 3.3.1 we get

$$E\xi_P(t) = \beta(t) + (f_o+m_o)\,\alpha(t)$$

$$- (f_o+m_o)\,\alpha(t)$$

$$- \beta(t)\, e^{-\beta(t)} [I_0(\beta(t))+I_1(\beta(t))]$$

$$+ G(\alpha(t), \beta(t); f_o, m_o)$$

or

$$E\xi_P(t) = \psi(\beta(t)) + G(\alpha(t), \beta(t); f_o, m_o).$$ QED

3.5 Asymptotic Distributions

In this section we study some properties of the distributions of the random variables $\xi_F(t)$, $\xi_M(t)$, $\xi(t)$, $\xi_{MS}(t)$, $\xi_{FS}(t)$, and $\xi_P(t)$ as $t \to \infty$. Throughout the section we assume that the limit of $Y(t)$ as $t \to \infty$ exists; we denote the limit by Y_∞.

From the definitions of the functions α and β in (3.2.1) and (3.2.2), respectively, we find that $\alpha(t) \to 0$ as $t \to \infty$, and, applying ℓ'Hospital's rule, that

$$\beta(t) \to \beta_\infty = \frac{\nu_1 Y_\infty}{\mu_1} \tag{3.5.1}$$

as $t \to \infty$.

From Lemma 3.2.1 it follows that

$$P\{\xi_F(t)=f\} \to \frac{(\frac{1}{2}\beta_\infty)^f}{f!} e^{-\frac{1}{2}\beta_\infty} \tag{3.5.2}$$

as $t \to \infty$. Hence the distribution of $\xi_F(t)$ approaches a Poisson distribution with parameter $\frac{1}{2}\beta_\infty$ as $t \to \infty$, independent of f_o. The distribution of $\xi_M(t)$ also approaches a Poisson distribution with parameter $\frac{1}{2}\beta_\infty$ as $t \to \infty$, while the distribution of $\xi(t)$ approaches a Poisson distribution with parameter β_∞. In particular, we find that

$$P\{\xi(t) > 0\} \to 1 - e^{-\beta_\infty} \tag{3.5.3}$$

as $t \to \infty$.

Applying Lemma 3.3.2, we find, for $k \geq 1$, that

$$P\{\xi_{MS}(t)=k\} \to e^{-\beta_\infty} I_k(\beta_\infty) \tag{3.5.4}$$

as $t \to \infty$.

In order to evaluate the limit value of $P\{\xi_{MS}(t) > 0\}$ as $t \to \infty$, we note from (3.3.9) that $I_n(x) < I_m(x)$ for $n > m \geq 0$ and $x > 0$. Let $k \geq m_o$. Then, by Lemma 3.3.2,

$$P\{\xi_{MS}(t) = k\} \leq I_{k-m_o}(\beta(t)).$$

Now the fact that the function Y is bounded implies that the function β is bounded. Indeed, let $Y(t) \leq M$ for all $t \geq 0$. Then $\beta(t) \leq \frac{\nu_1 M}{\mu_1}$ for all $t \geq 0$. The functions $I_n(x)$ are increasing in x for $x \geq 0$ and fixed n, as deduced from (3.3.4). Therefore $I_{k-m_o}(\beta(t)) \leq I_{k-m_o}\left(\frac{\nu_1 M}{\mu_1}\right)$, $t \geq 0$. But

$$\sum_{k=m_o}^{\infty} I_{k-m_o}\left(\frac{\nu_1 M}{\mu_1}\right)$$

converges by (3.3.5). We conclude that $\sum_{k=1}^{\infty} P\{\xi_{MS}(t) = k\}$ converges uniformly for $t \geq 0$. Hence we have, applying (3.5.4) and (3.3.5), that

$$P\{\xi_{MS}(t) > 0\} \to \tfrac{1}{2}\,(1-e^{-\beta_\infty} I_0(\beta_\infty)) \tag{3.5.5}$$

as $t \to \infty$. The distributions of the two random variables $\xi_{FS}(t)$ and $\xi_{MS}(t)$ approach the same limit as $t \to \infty$.

From the definition of the function G in Section 3.3, it follows that

$$G(\alpha(t), \beta(t); f_o, m_o) \to 0 \text{ as } t \to \infty.$$

Hence, using Theorem 3.3.1, we conclude that

$$E\xi_{MS}(t) \to \frac{1}{2} \beta_\infty e^{-\beta_\infty} (I_0(\beta_\infty)+I_1(\beta_\infty)) \tag{3.5.6}$$

as $t \to \infty$. The expectation of the random variable $\xi_{FS}(t)$ has the same limit value.

Relation (3.4.1) shows that

$$P\{\xi_P(t) > 0\} \to \left(1-e^{-\frac{1}{2}\beta_\infty}\right)^2 \tag{3.5.7}$$

as $t \to \infty$, while Theorem 3.4.1 implies that

$$E\xi_P(t) \to \psi(\beta_\infty) \tag{3.5.8}$$

as $t \to \infty$.

The results in (3.5.7) and (3.5.8) were given by Macdonald [14], although not in explicit form.

3.6 Some Properties of the Function G

The function G which was defined in Section 3.3 will play a role in the development of Chapter 8. As a preparation for this, we study some properties of G and of its derivatives with respect to α and β. Throughout the section we assume that $0 \leq \alpha \leq 1$, $0 \leq \beta$, and that f_o, m_o are non-negative integers.

Lemma 3.6.1

(i) $G(0,\beta;f_o,m_o) = 0$;

(ii) $G(\alpha,\beta;0,0) = 0$;

(iii) $G(\alpha,\beta;f_o,m_o) \geq 0$;

(iv) $G(\alpha,0;f_o,m_o) = 0$ if $f_o \wedge m_o = 0$;

(v) $G(\alpha,0;f_o,m_o) > 0$ if $\alpha > 0$, $f_o \wedge m_o > 0$;

(vi) $G(\alpha,\beta;f_o,m_o) = 0(\alpha^2) + 0(\alpha\beta)$ $(\alpha\to 0,\ \beta\to 0)$;

(vii) $G_\alpha(0,\beta;f_o,m_o) = (f_o+m_o)\psi'(\beta)$.

Proof: The first two assertions are verified directly from the definition of G in (3.3.16).

From the definitions of g_1 and g_2 in (3.3.11) and (3.3.12), respectively, we find

$$g_1(\alpha,\beta;f_o,m_o) \leq m_o\,\alpha \sum_{k=1}^{m_o-1} e^{-\beta} I_k(\beta)$$

and

$$g_2(\alpha,\beta;f_o,m_o) \leq m_o\,\alpha \left[e^{-\beta} I_0(\beta) + 2 \sum_{k=1}^{m_o-1} e^{-\beta} I_k(\beta) \right].$$

An application of (3.3.5) then gives

$$g_2(\alpha,\beta;f_o,m_o) \leq m_o\,\alpha$$

and

$$g_2(\alpha,\beta;m_o,f_o) \leq f_o\,\alpha.$$

The definition of G in (3.3.16) leads to the inequality

$$G(\alpha,\beta;f_o,m_o) \geq 0.$$

Now, let $f_o \wedge m_o = 0$. From their definitions in (3.3.11) and (3.3.14) we find $g_1(\alpha,0;f_o,m_o)$ $= g_4(\alpha,0;f_o,m_o) = 0$. Hence, by Lemma 3.3.3, $g_3(\alpha,0;f_o,m_o)$ $= (f_o+m_o)\alpha$. Therefore we find, using Lemma 3.3.4, that $G(\alpha,0;f_o,m_o) = 0$.

Next, let $f_o \wedge m_o > 0$ and $\alpha > 0$. From their definitions in (3.3.11) and (3.3.14), respectively, we find $g_1(\alpha,0;f_o,m_o) = 0$ and $g_4(\alpha,0;f_o,m_o) > 0$. Combining Lemmas 3.3.3 and 3.3.4 we get $G(\alpha,0;f_o,m_o) = 2g_4(\alpha,0;f_o,m_o) > 0$.

From (3.4.5) we have $e^{-\beta}I_k(\beta) = 0(\beta^k)$ as $\beta\to 0$ and $1 - e^{-\beta}I_0(\beta) = 0(\beta)$ as $\beta\to 0$. Hence, using (3.3.11), $g_1(\alpha,\beta;f_o,m_o) = 0(\alpha^2\beta)$ as $\alpha\to 0$, $\beta\to 0$. Furthermore, from (3.3.14), $g_4(\alpha,\beta;f_o,m_o) = 0(\alpha^2)$ as $\alpha\to 0$, $\beta\to 0$. Assertions (vi) and (vii) now follow by combining Lemmas 3.3.3 and 3.3.4, and applying (3.4.3) for (vii). QED

We now proceed to study the derivative of G with respect to β.

Lemma 3.6.2

$$\frac{\partial}{\partial\beta} g_1(\alpha,\beta;f_o,m_o)$$

$$= \frac{1}{2} e^{-\beta}I_0(\beta) \sum_{i=0}^{f_o} \sum_{j=i+1}^{m_o} (j-i-1)\binom{f_o}{i}\binom{m_o}{j} \alpha^{i+j}(1-\alpha)^{f_o+m_o-i-j}$$

$$- \frac{1}{2} e^{-\beta}I_1(\beta) \sum_{i=0}^{f_o} \sum_{j=i+2}^{m_o} (j-i)\binom{f_o}{i}\binom{m_o}{j} \alpha^{i+j}(1-\alpha)^{f_o+m_o-i-j}$$

$$+ \frac{1}{2} \sum_{i=0}^{f_o} \sum_{j=i+2}^{m_o} e^{-\beta}I_{j-i}(\beta)\binom{f_o}{i}\binom{m_o}{j} \alpha^{i+j}(1-\alpha)^{f_o+m_o-i-j}.$$

Proof: From relations (3.3.4) and (3.3.2) we get

$$\frac{d}{d\beta}[e^{-\beta}I_k(\beta)] = e^{-\beta}[\tfrac{1}{2}\, I_{k-1}(\beta)-I_k(\beta)+\tfrac{1}{2}\, I_{k+1}(\beta)]$$

and

$$\frac{d}{d\beta}[e^{-\beta}I_0(\beta)] = - e^{-\beta}[I_0(\beta)-I_1(\beta)].$$

The definition (3.3.11) of the function g_1 gives

$$\frac{\partial}{\partial\beta}\, g_1(\alpha,\beta;f_o,m_o)$$

$$= \frac{1}{2} \sum_{k=0}^{m_o-2} e^{-\beta}I_k(\beta) \sum_{i=0}^{f_o} \sum_{j=k+i+1}^{m_o} (j-k-i-1) \binom{f_o}{i}\binom{m_o}{j}\, \alpha^{i+j}(1-\alpha)^{f_o+m_o-i-j}$$

$$- \sum_{k=1}^{m_o-1} e^{-\beta}I_k(\beta) \sum_{i=0}^{f_o} \sum_{j=k+i+1}^{m_o} (j-k-i) \binom{f_o}{i}\binom{m_o}{j}\, \alpha^{i+j}(1-\alpha)^{f_o+m_o-i-j}$$

$$+ \frac{1}{2} \sum_{k=2}^{m_o} e^{-\beta}I_k(\beta) \sum_{i=0}^{f_o} \sum_{j=k+i+1}^{m_o} (j-k-i+1) \binom{f_o}{i}\binom{m_o}{j}\, \alpha^{i+j}(1-\alpha)^{f_o+m_o-i-j}$$

$$+ \frac{1}{2} \sum_{k=2}^{m_o} e^{-\beta}I_k(\beta) \sum_{i=0}^{f_o} \binom{f_o}{i}\binom{m_o}{k+i}\, \alpha^{2i+k}(1-\alpha)^{f_o+m_o-2i-k}.$$

The last sum here was added in order to make the j-summation in the next to last sum run from k + i + 1 to m_o. The above expression is simplied by first collecting terms that contain the factor j - k - i, and then combining those terms containing this factor for which k runs from 2 to m_o - 2. After several cancellations we get

$$\frac{\partial}{\partial\beta} g_1(\alpha,\beta;f_o,m_o)$$

$$= \frac{1}{2} e^{-\beta} I_0(\beta) \sum_{i=0}^{f_o} \sum_{j=i+1}^{m_o} (j-i-1) \binom{f_o}{i}\binom{m_o}{j} \alpha^{i+j}(1-\alpha)^{f_o+m_o-i-j}$$

$$- \frac{1}{2} e^{-\beta} I_1(\beta) \sum_{i=0}^{f_o} \sum_{j=i+2}^{m_o} (j-i) \binom{f_o}{i}\binom{m_o}{j} \alpha^{i+j}(1-\alpha)^{f_o+m_o-i-j}$$

$$+ \frac{1}{2} \sum_{k=2}^{m_o} e^{-\beta} I_k(\beta) \sum_{i=0}^{f_o} \binom{f_o}{i}\binom{m_o}{k+i} \alpha^{2i+k}(1-\alpha)^{f_o+m_o-2i-k}.$$

The result now follows by interchanging order of summation in the last sum and replacing the summation variable k by $j = k + i$. QED

Lemma 3.6.3

$$\frac{\partial}{\partial\beta} g_2(\alpha,\beta;f_o,m_o)$$

$$= - e^{-\beta} I_0(\beta) \sum_{i=0}^{f_o} \sum_{j=i+1}^{m_o} \binom{f_o}{i}\binom{m_o}{j} \alpha^{i+j}(1-\alpha)^{f_o+m_o-i-j}$$

$$+ \sum_{i=0}^{f_o} \sum_{j=i+1}^{m_o} e^{-\beta} I_{j-i}(\beta) \binom{f_o}{i}\binom{m_o}{j} \alpha^{i+j}(1-\alpha)^{f_o+m_o-i-j}.$$

Proof: The definition (3.3.12) of the function g_2 gives

$$\frac{\partial}{\partial\beta}\, g_2(\alpha,\beta;f_o,m_o)$$

$$= - e^{-\beta}[I_0(\beta)-I_1(\beta)] \sum_{i=0}^{f_o} \sum_{j=i+1}^{m_o} (j-i) \binom{f_o}{i}\binom{m_o}{j} \alpha^{i+j}(1-\alpha)^{f_o+m_o-i-j}$$

$$+ 2\, \frac{\partial}{\partial\beta}\, g_1(\alpha,\beta;f_o,m_o).$$

The result follows by inserting the expression for the derivative of g_1 with respect to β from Lemma 3.6.2. QED

Lemma 3.6.4

$$\frac{\partial}{\partial\beta}\, G(\alpha,\beta;f_o,m_o) = e^{-\beta} I_0(\beta)$$

$$- \sum_{i=0}^{f_o} \sum_{j=0}^{m_o} e^{-\beta} I_{j-i}(\beta) \binom{f_o}{i}\binom{m_o}{j} \alpha^{i+j}(1-\alpha)^{f_o+m_o-i-j}.$$

Proof: The definition (3.3.16) of the function G gives

$$\frac{\partial}{\partial\beta}\, G(\alpha,\beta;f_o,m_o)$$

$$= - \frac{\partial}{\partial\beta}\, g_2(\alpha,\beta;f_o,m_o) - \frac{\partial}{\partial\beta}\, g_2(\alpha,\beta;m_o,f_o).$$

By using Lemma 3.6.3 and the identity

$$\sum_{i=0}^{f_o} \sum_{j=i+1}^{m_o} \gamma_{ij} + \sum_{j=0}^{m_o} \sum_{i=j+1}^{f_o} \gamma_{ij}$$

$$= \sum_{i=0}^{f_o} \sum_{j=0}^{m_o} \gamma_{ij} - \sum_{i=0}^{f_o \wedge m_o} \gamma_{ii} \quad \text{we get}$$

$$\frac{\partial}{\partial\beta} G(\alpha,\beta;f_o,m_o)$$

$$= e^{-\beta} I_0(\beta)$$

$$- e^{-\beta} I_0(\beta) \sum_{i=0}^{f_o \wedge m_o} \binom{f_o}{i}\binom{m_o}{i} \alpha^{2i}(1-\alpha)^{f_o+m_o-2i}$$

$$- \sum_{i=0}^{f_o} \sum_{j=0}^{m_o} e^{-\beta} I_{j-i}(\beta) \binom{f_o}{i}\binom{m_o}{j} \alpha^{i+j}(1-\alpha)^{f_o+m_o-i-j}$$

$$+ \sum_{i=0}^{f_o \wedge m_o} e^{-\beta} I_0(\beta) \binom{f_o}{i}\binom{m_o}{i} \alpha^{2i}(1-\alpha)^{f_o+m_o-2i} .$$

The result now follows after a cancellation of the second and fourth terms. QED

The following lemma shows that the derivative of G with respect to β is non-negative.

Lemma 3.6.5

$$\frac{\partial}{\partial\beta} G(\alpha,\beta;f_o,m_o) \geq 0.$$

Proof: Apply (3.3.9) in Lemma 3.6.4. QED

We now proceed to study the derivative of G with respect to α. Let $k \geq 0$ be an integer and define

$$h(\alpha;k,f_o,m_o)$$

$$= \sum_{i=0}^{f_o} \sum_{j=k+i+1}^{m_o} (j-k-i) \binom{f_o}{i} \binom{m_o}{j} \alpha^{i+j} (1-\alpha)^{f_o+m_o-i-j}. \qquad (3.6.1)$$

Note that

$$h(\alpha;k,0,m_o) = \sum_{j=k+1}^{m_o} (j-k) \binom{m_o}{j} \alpha^j (1-\alpha)^{m_o-j}, \qquad (3.6.2)$$

and hence

$$h(\alpha;k,f_o,m_o) = \sum_{i=0}^{f_o} \binom{f_o}{i} \alpha^i (1-\alpha)^{f_o-i} h(\alpha;k+i,0,m_o). \qquad (3.6.3)$$

Lemma 3.6.6

$$\frac{d}{d\alpha} h(\alpha;k,0,m_o) = m_o \sum_{j=k}^{m_o-1} \binom{m_o-1}{j} \alpha^j (1-\alpha)^{m_o-1-j}.$$

Proof:

Note first that the lemma holds for k = 0. From (3.6.2) we have

$$h(\alpha;0,0,m_o) = m_o \alpha,$$

and hence

$$\frac{d}{d\alpha} h(\alpha;0,0,m_o) = m_o,$$

as asserted in the lemma.

For $k \geq 1$ we make use of the incomplete beta-function I_α. The following two relations are derived from Abramowitz-Stegun [1].

$$I_\alpha(k+1,m_o-k) = \sum_{j=k+1}^{m_o} \binom{m_o}{j} \alpha^j (1-\alpha)^{m_o-j}; \qquad (3.6.4)$$

$$I_\alpha(k+1,m_o-k) = \frac{m_o!}{k!(m_o-k-1)!} \int_0^\alpha t^k (1-t)^{m_o-k-1} dt. \qquad (3.6.5)$$

From (3.6.2), (3.6.4), and (3.6.5) we get

$$h(\alpha;k,0,m_o) = m_o \alpha I_\alpha(k,m_o-k) - kI_\alpha(k+1,m_o-k)$$

$$= \frac{m_o!\alpha}{(k-1)!(m_o-k-1)!} \int_0^\alpha t^{k-1}(1-t)^{m_o-k-1} dt$$

$$- \frac{m_o!}{(k-1)!(m_o-k-1)!} \int_0^\alpha t^k (1-t)^{m_o-k-1} dt.$$

Hence

$$\frac{d}{d\alpha} h(\alpha;k,0,m_o) = m_o I_\alpha(k,m_o-k). \qquad \text{QED}$$

The following lemma gives an expression for the derivative of h with respect to α.

Lemma 3.6.7

$$\frac{d}{d\alpha} h(\alpha;k,f_o,m_o) = \frac{d}{d\alpha} h(\alpha;k,0,m_o)$$

$$- m_o \sum_{i=1}^{f_o} \sum_{j=k}^{k+i-1} \binom{f_o}{i}\binom{m_o-1}{j} \alpha^{i+j} (1-\alpha)^{f_o+m_o-i-j-1}$$

$$- \sum_{i=1}^{f_o} \sum_{j=k+i}^{m_o} i \binom{f_o}{i}\binom{m_o}{j} \alpha^{i+j-1} (1-\alpha)^{f_o+m_o-i-j}.$$

Proof: From expression (3.6.3) we can write

$$\frac{d}{d\alpha} h(\alpha;k,f_o,m_o) = S_1 + S_2,$$

where

$$S_1 = \sum_{i=0}^{f_o} \frac{d}{d\alpha} \left[\binom{f_o}{i} \alpha^i (1-\alpha)^{f_o-i} \right] h(\alpha;k+i,0,m_o)$$

and

$$S_2 = \sum_{i=0}^{f_o} \binom{f_o}{i} \alpha^i (1-\alpha)^{f_o-i} \frac{d}{d\alpha} h(\alpha;k+i,0,m_o).$$

Now

$$S_1 = \sum_{i=1}^{f_o} i \binom{f_o}{i} \alpha^{i-1}(1-\alpha)^{f_o-i} h(\alpha;k+i,0,m_o)$$

$$- \sum_{i=0}^{f_o-1} (f_o-i) \binom{f_o}{i} \alpha^{i}(1-\alpha)^{f_o-i-1} h(\alpha,k+i,0,m_o).$$

By using the identity

$$(f_o-i) \binom{f_o}{i} = (i+1) \binom{f_o}{i+1}$$

we get

$$S_1 = \sum_{i=1}^{f_o} i \binom{f_o}{i} \alpha^{i-1}(1-\alpha)^{f_o-i} [h(\alpha;k+i,0,m_o)-h(\alpha;k+i-1,0,m_o)].$$

Now relation (3.6.2) gives

$$h(\alpha;k+i,0,m_o) - h(\alpha;k+i-1,0,m_o)$$

$$= - \sum_{j=k+i}^{m_o} \binom{m_o}{j} \alpha^{j}(1-\alpha)^{m_o-j}.$$

Hence

$$S_1 = - \sum_{i=1}^{f_o} \sum_{j=k+i}^{m_o} i \binom{f_o}{i}\binom{m_o}{j} \alpha^{i+j-1}(1-\alpha)^{f_o+m_o-i-j}.$$

Next, for evaluating S_2, we apply Lemma 3.6.6 and observe that

$$\frac{d}{d\alpha} h(\alpha;k+i,0,m_o)$$

$$= \frac{d}{d\alpha} h(\alpha;k,0,m_o) - m_o \sum_{j=k}^{k+i-1} \binom{m_o-1}{j} \alpha^j (1-\alpha)^{m_o-j-1}.$$

Insertion of this expression into S_2 and addition of the above expression for S_1 gives the required result. QED

Lemma 3.6.8

$$\frac{d}{d\alpha} h(\alpha;k,f_o,m_o) \leq m_o.$$

Proof: From Lemma 3.6.6 we have

$$\frac{d}{d\alpha} h(\alpha;k,0,m_o) \leq m_o.$$

From Lemma 3.6.7 we find

$$\frac{d}{d\alpha} h(\alpha;k,f_o,m_o) \leq \frac{d}{d\alpha} h(\alpha;k,0,m_o). \qquad \text{QED}$$

Lemma 3.6.9

$$\frac{\partial}{\partial\alpha} G(\alpha,\beta;f_o,m_o) \geq 0.$$

If $\beta > 0$ and $f_o \vee m_o \geq 1$, then

$$\frac{\partial}{\partial\alpha} G(\alpha,\beta;f_o,m_o) > 0.$$

Proof:

Using expression (3.3.16) for the function G and (3.6.1) for the function h we get

$$\frac{\partial}{\partial\alpha} G(\alpha,\beta;f_o,m_o)$$

$$= f_o + m_o$$

$$- e^{-\beta} I_0(\beta) \frac{d}{d\alpha} h(\alpha;0,f_o,m_o)$$

$$- 2 \sum_{k=1}^{m_o-1} e^{-\beta} I_k(\beta) \frac{d}{d\alpha} h(\alpha;k,f_o,m_o)$$

$$- e^{-\beta} I_0(\beta) \frac{d}{d\alpha} h(\alpha;0,m_o,f_o)$$

$$- 2 \sum_{k=1}^{f_o-1} e^{-\beta} I_k(\beta) \frac{d}{d\alpha} h(\alpha;k,m_o,f_o).$$

From Lemma 3.6.8 we get

$$\frac{\partial}{\partial\alpha} G(\alpha,\beta;f_o,m_o)$$

$$\geq f_o \left[1-e^{-\beta}I_0(\beta) - 2 \sum_{k=1}^{f_o-1} e^{-\beta}I_k(\beta)\right]$$

$$+ m_o \left[1-e^{-\beta}I_0(\beta) - 2 \sum_{k=1}^{m_o-1} e^{-\beta}I_k(\beta)\right].$$

From relation (3.3.5) we conclude

$$\frac{\partial}{\partial\alpha} G(\alpha,\beta;f_o,m_o) \geq 0.$$

If $\beta > 0$ and $f_o \vee m_o \geq 1$, the inequality is strict. QED

Chapter 4

Model for Intermediate Hosts

4.1 Model Formulation

In this chapter we establish a model for the number of infected intermediate hosts. The total population of intermediate hosts is assumed constant; we denote its size by the integer $N_2 > 0$. We use the random variable $\eta(t)$ to denote the number of infected intermediate hosts at time t, with $t \geq 0$.

As model for the intermediate hosts we propose a continuous time, finite state, Markov process, whose transition probabilities satisfy the conditions given below. We refer to this process as a non-homogeneous infection-recovery process.

The transition probability functions are denoted by

$$\Pi_{ij}(t,u) = P\{\eta(u)=j \mid \eta(t)=i\},\ i,j = 0,1,\ldots,N_2;\ 0 \leq t < u.$$

We postulate the following forms of these functions for $u - t = \Delta t$ as $\Delta t \to 0$:

$$\Pi_{i,i+1}(t,t+\Delta t) = \nu_2 X(t)(N_2-i)\Delta t + o(\Delta t),$$

$$\Pi_{i,i-1}(t,t+\Delta t) = i\mu_2\Delta t + o(\Delta t),$$

$$\Pi_{i,i}(t,t+\Delta t) = 1 - [\nu_2 X(t)(N_2-i) + i\mu_2]\Delta t + o(\Delta t);$$

here, $\nu_2 > 0$ and $\mu_2 > 0$ are constants, while $X(t) \geq 0$ is a bounded continuous function defined for $t \geq 0$. We refer to μ_2 as the recovery rate per infected intermediate host, and to $\nu_2 X(t)$ as the infection rate per uninfected intermediate host. We assume that $\eta(0) = N_2 y_o = m_2$, where $0 \leq y_o \leq 1$. Hence y_o is the initial proportion of intermediate hosts that are infected.

The probability distribution of $\eta(t)$ is denoted by

$$\Pi_i(t) = P\{\eta(t)=i\} = \Pi_{m_2,i}(0,t), \quad i=0,1,\ldots,N_2.$$

Using the above infection rate and recovery rate, the following differential-difference equations (Kolmogoroff forward equations) are established for the functions $\Pi_i(t)$:

$$\begin{aligned} \Pi_i'(t) = {} & \nu_2 X(t)(N_2-i+1)\, \Pi_{i-1}(t) \\ & - [\nu_2 X(t)(N_2-i) + i\mu_2]\, \Pi_i(t) \\ & + (i+1)\, \mu_2 \Pi_{i+1}(t), \end{aligned} \tag{4.1.1}$$

$i = 0,1,\ldots,N_2$, $t \geq 0$; here, we define $\Pi_{-1}(t) \equiv \Pi_{N_2+1}(t) \equiv 0$. At time $t = 0$ we have

$$\Pi_i(0) = \begin{cases} 1, & i = m_2, \\ 0, & i \neq m_2. \end{cases}$$

The infection rate and recovery rate specified above lead to a unique family of transition probability functions for the Markov process $(\eta(t))_{t\geq 0}$. This follows since these functions satisfy a linear system of ordinary differential equations of finite order with continuous coefficients. In particular, it follows that the above system of differential-difference equations for the functions $\Pi_i(t)$ has a unique solution.

4.2 The Distribution of Infected Intermediate Hosts

We introduce the following notation:

$$\rho(t) = \nu_2 \int_0^t X(s)ds + \mu_2 t, \tag{4.2.1}$$

$$\sigma(t) = \nu_2 \int_0^t X(s)e^{\rho(s)}ds, \tag{4.2.2}$$

$$p(t) = e^{-\rho(t)}\,\sigma(t), \tag{4.2.3}$$

$$\bar{p}(t) = e^{-\rho(t)}\,[\sigma(t)+1]. \tag{4.2.4}$$

Note that

$$\rho'(t) = \nu_2 X(t) + \mu_2.$$

Hence

$$p(t) \leq e^{-\rho(t)} \int_0^t \rho'(s)\, e^{\rho(s)} ds = 1 - e^{-\rho(t)}.$$

We therefore conclude that the inequalities $0 \leq p(t) \leq 1$ and $0 \leq \bar{p}(t) \leq 1$ hold for all $t \geq 0$.

The following lemma gives the distribution and expectation of $\eta(t)$, the number of infected intermediate hosts at time t.

Lemma 4.2.1

$$P\{\eta(t) = k\}$$

$$= \sum_{j=0}^{m_2} \binom{m_2}{j} (\bar{p}(t))^j (1-\bar{p}(t))^{m_2-j} \binom{N_2-m_2}{k-j} (p(t))^{k-j} (1-p(t))^{N_2-m_2-k+j}$$

and

$$E\eta(t) = N_2 p(t) + m_2 e^{-\rho(t)}.$$

Proof: We introduce the probability generating function

$$H(t,z) = \sum_{i=0}^{N_2} \Pi_i(t) z^i,$$

where the functions $\Pi_i(t)$ satisfy the differential-difference equations of the preceding section. Then $H(t,z)$ satisfies the partial differential equation

$$\frac{\partial H(t,z)}{\partial t} + (z-1)(\nu_2 X(t)z+\mu_2)\frac{\partial H(t,z)}{\partial z}$$

$$= \nu_2 N_2 X(t)(z-1)H(t,z),$$

with initial data specified along the line $t = 0$ by $H(0,z) = z^{m_2}$. The characteristics are solutions of the Riccati equation

$$\frac{dz}{dt} = (z-1)(\nu_2 X(t)z+\mu_2).$$

Any solution of this equation can be written

$$z(t) = 1 + \frac{e^{\rho(t)}}{K-\sigma(t)},$$

where K is a constant. At $t = 0$ we have

$$z(0) = 1 + \frac{1}{K}.$$

To solve the above initial-value problem for $H(t,z)$, we consider the ordinary differential equation

$$H'(t) = N_2 \frac{\sigma'(t)}{K-\sigma(t)} H(t)$$

with initial condition

$$H(0) = \left(1 + \frac{1}{K}\right)^{m_2}.$$

This is the relation to be satisfied by H along a characteristic determined by K. The solution is

$$H(t) = \left(1 + \frac{1}{K}\right)^{m_2} \left[\frac{K}{K-\sigma(t)}\right]^{N_2}.$$

The solution of the initial-value problem posed above for the function H can therefore be written

$$H(t,z) = H_1(t,z)\, H_2(t,z),$$

where

$$H_1(t,z) = [1 + \bar{p}(t)(z-1)]^{m_2}$$

and

$$H_2(t,z) = [1 + p(t)(z-1)]^{N_2-m_2}.$$

We recognize $H_1(t,z)$ to be the probability generating function of a random variable $\eta_1(t)$, whose distribution is binomial with parameters m_2 and $\bar{p}(t)$. Likewise, $H_2(t,z)$ is the probability generating function of a random variable $\eta_2(t)$, again binomially distributed with parameters N_2-m_2 and $p(t)$. It follows that $\eta(t)$ can be expressed as the sum of the independent random variables $\eta_1(t)$ and $\eta_2(t)$:

$$\eta(t) = \eta_1(t) + \eta_2(t).$$

The distribution of $\eta(t)$ is now found by convolving the distributions of $\eta_1(t)$ and $\eta_2(t)$, while the expectation of $\eta(t)$ is the sum of the expectations of $\eta_1(t)$ and $\eta_2(t)$.

QED

We note that $E\eta(t)$ satisfies the linear ordinary differential equation

$$\frac{d}{dt} E\eta(t) = \nu_2 X(t) [N_2 - E\eta(t)] - \mu_2 E\eta(t) \tag{4.2.5}$$

with

$$E\eta(0) = N_2 y_o. \tag{4.2.6}$$

4.3 The Asymptotic Distribution

In this section we assume that $\lim_{t\to\infty} X(t) = X_\infty$ exists. Then we find by an application of ℓ'Hospital's rule that

$$\lim_{t\to\infty} p(t) = \lim_{t\to\infty} \bar{p}(t) = \frac{\nu_2 X_\infty}{\nu_2 X_\infty + \mu_2}.$$

It follows that the distributions of $\eta_1(t)$ and $\eta_2(t)$ approach binomial distributions with the same probability of success as $t \to \infty$. Therefore, $\eta(t)$ converges (in distribution) to a binomially distributed random variable with parameters N_2 and $\frac{\nu_2 X_\infty}{\nu_2 X_\infty + \mu_2}$ as $t \to \infty$. The expected value of $\eta(t)$ approaches the value $\frac{N_2 \nu_2 X_\infty}{\nu_2 X_\infty + \mu_2}$ as $t \to \infty$.

Chapter 5

Some Results From the Theory of Differential Equations

The aim of this chapter is to summarize certain definitions and results concerning solutions to autonomous systems of ordinary differential equations. The concepts and results discussed here find repeated applications in Chapters 6 and 8.

We introduce the following notation:

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \text{ is a vector in Euclidean n-space,}$$

$$v(t) = \begin{pmatrix} v_1(t) \\ \vdots \\ v_n(t) \end{pmatrix} \text{ is a vector-valued function defined}$$

on some interval of the real line,

$$g(v) = \begin{pmatrix} g_1(v) \\ \vdots \\ g_n(v) \end{pmatrix} \text{ is a vector-valued function defined}$$

throughout Euclidean n-space. In our applications, n takes the values 2 or 3.

We use D to denote a subset of Euclidean n-space. The interior of the set D is denoted by $\overset{\circ}{D}$, its closure by $\overline{D}$, and its boundary by ∂D. Throughout the discussion we restrict ourselves to sets D whose boundary ∂D is the union of finitely many smooth simply connected surfaces in case n = 3, respectively curves in case n = 2. Hence the boundary ∂D is smooth at all points except possibly those that belong to the intersection of two or more surfaces or curves.

We assume that the Jacobian matrix $J = \frac{\partial g}{\partial v}$ (with components $\frac{\partial g_i}{\partial v_j}$, $i,j = 1,2,\ldots,n$) exists and is continuous for all v.

Now consider the initial-value problem

$$\begin{cases} v' = g(v) \\ v(0) = v_o, \end{cases} \qquad (P)$$

where v_o is arbitrary. The Picard-Lindelöf Theorem then applies (Hartman [11], page 8). It asserts local existence and uniqueness of the solution of problem (P), i.e., it asserts the existence of $\varepsilon > 0$ such that problem (P) has a unique solution v(t) in the interval $-\varepsilon < t < \varepsilon$.

In the sequel we are concerned with global existence, i.e., existence of a solution to problem (P) for all $t \geq 0$. In this context we make use of the following definitions:

Definition 5.1

A set D is called an invariant set with respect to the system $v' = g(v)$ if for each point $v_o \in D$ the solution $v(t)$ of Problem (P) exists for all $t \geq 0$, is uniquely determined, and $v(t) \in D$ for all $t \geq 0$.

Definition 5.2

A point $v_o \in \partial D$ is called an ingress point of D with respect to the system $v' = g(v)$ if corresponding to the solution $v(t)$ of Problem (P) there exists $\varepsilon > 0$ such that $v(t) \in D$ for all t in the interval $0 < t < \varepsilon$.

Proposition 5.1

(i) Let $A = D_1 \cap D_2$ and let $v_o \in \partial A$. Assume that $v_o \in \partial D_1 - \partial D_2$ and that v_o is an ingress point of D_1. Then v_o is an ingress point of A.

(ii) Let $A = D_1 \cap D_2$ and let $v_o \in \partial A$. Assume that $v_o \in \partial D_1 \cap \partial D_2$ and that v_o is an ingress point of D_1 and D_2. Then v_o is an ingress point of A.

(iii) Let $A = D_1 \cap D_2$. Assume that all boundary points of D_i that belong to $\bar{A}$ are ingress points of D_i, $i = 1, 2$. Then all boundary points of A are ingress points of A.

Proof:

Apply Definition 5.2 in the proof of (i) and (ii). Apply (i) and (ii) in the proof of (iii).

Proposition 5.2

Let D be a bounded set and assume that every boundary point of D is an ingress point of $\bar{D}$. Then $\bar{D}$ is an invariant set.

Proof: Local existence and uniqueness follow from the assumed continuity of J.

Let E be a bounded open set such that $\bar{D} \subset E$. Then Corollary 3.2 of Hartman [11], page 14, applies since g is continuous on $\bar{E}$. It follows that the solution v(t) can be extended over a t-interval I, where either $I = [0, \infty)$ or $I = [0, \tau]$ with $0 < \tau < \infty$ and $v(\tau) \in \partial E$. We show that the latter case leads to a contradiction. Let

$$B = \{t : 0 \leq t \leq \tau, v(t) \in \partial D\}.$$

The set B is closed since ∂D is closed and v is continuous. Hence B has a largest element t_o. Now the boundaries ∂D and ∂E are disjoint and $v(\tau) \in \partial E$. Hence $t_o < \tau$. It follows that $v(t) \notin \bar{D}$ for $t_o < t < \tau$. We are thus led to the contradiction that $v(t_o)$ is not an ingress point of $\bar{D}$.

Therefore, we conclude that $I = [0, \infty)$ and that $v(t) \in \bar{D}$ for all $t \geq 0$.

Global uniqueness of the solution follows from global existence and local uniqueness. QED

The next proposition gives a sufficient condition for a boundary point of a set D to be an ingress point of $\overset{o}{D}$. We let v_o denote a point on the boundary ∂D, and we assume that $\partial D \in C^1$ in some neighborhood of v_o. We use N to denote a normal to ∂D at v_o, where N is pointing into the set D. We recall that if $\phi(x,y,z) = 0$ is the equation of a surface (n=3), then the gradient (ϕ_x, ϕ_y, ϕ_z) is a normal to the surface. Furthermore, we let T(t) dencte the tangent to the solution v(t) of problem (P), defined by $T(t) = (v_1'(t),\ldots,v_n'(t))$.

Proposition 5.3 With the above notation, assume that the inner product $N \cdot T(0) > 0$. Then the point v_o is an ingress point of the set $\overset{\circ}{D}$ with respect to the system $v' = g(v)$.

Proof: Assume the proposition is not true. Then there exists a decreasing sequence $\{t_n\}$ of positive numbers such that $t_n \to 0$ and $v(t_n) \notin \overset{\circ}{D}$, $n = 1,2,\ldots$ For each n, consider the vector relation

$$\frac{v(t_n)-v_o}{t_n} = \frac{1}{t_n}\int_0^{t_n} v'(s)\,ds$$

$$= \frac{1}{t_n}\int_0^{t_n} g(v(s))\,ds = g(v(t_n^*)),$$

where $0 < t_n^* < t_n$. Hence

$$\frac{v(t_n)-v_o}{t_n} \cdot N = g(v(t_n^*)) \cdot N.$$

The left-hand side of this relation is non-positive for sufficiently large n. Now let $n \to \infty$. The limit of the left-hand side is non-positive, while the limit of the right-hand side is $N \cdot T(0)$, which is positive by hypothesis. This contradiction proves the truth of the proposition. QED

We make the observation that if $N \cdot T(0) < 0$, then the point v_o is an ingress point of the interior of the complement of the set D.

The following basic property of solutions to an autonomous system of differential equations is important. Let $v_1(t)$, $t \geq 0$, be the solution to problem (P). Furthermore, let $t_o > 0$, and let $v_2(t)$, $t \geq 0$, be the solution to problem (P) with $v_o = v_1(t_o)$. Then the trajectory traced in phase space by the solution $v_1(t)$ for $t \geq t_o$ is identical to the trajectory traced by $v_2(t)$ for $t \geq 0$.

In the sequel we study the behavior of solutions of problem (P) as $t \to \infty$. In this study we make use of the notions of critical point and of stability and asymptotic stability in the sense of Lyapunov. A point $\bar{v}$ is said to be a critical point of the system $v' = g(v)$ if it satisfies $g(\bar{v}) = 0$. Stability definitions of critical points are given in the following definition.

Definition 5.3 A critical point $\bar{v}$ of the system $v' = g(v)$ is said to be stable if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that the initial-value problem

$$\begin{cases} v' = g(v) \\ v(0) = v_o, \end{cases}$$

with $|v_o - \bar{v}| < \delta$, has a solution $v(t)$ which exists for all $t \geq 0$, and satisfies

$$|v(t) - \bar{v}| < \varepsilon$$

for all $t \geq 0$.

The critical point $\bar{v}$ is said to be asymptotically stable if it is stable and in addition there exists $\delta_o > 0$ such that if $v(t)$ is a solution of the above initial-value problem with $|v_o - \bar{v}| < \delta_o$, then

$$|v(t) - \bar{v}| \to 0$$

as $t \to \infty$.

The critical point $\bar{v}$ is said to be unstable if it is not stable.

The next proposition gives sufficient conditions for asymptotic stability and instability, respectively, in case the zero vector is a critical point.

Proposition 5.4 Consider the system

$$v' = Av + b(v),$$

where A is a constant matrix, and $|b(v)| = o(|v|)$ as $v \to 0$. If all the eigenvalues of A have negative real parts, then the identically zero solution is asymptotically stable. If, on the other hand, at least one of the eigenvalues of A has positive real part, then the identically zero solution is not stable.

Proof: See Coddington-Levinson [6], pages 314-315 and 317-318.

The behavior of solutions in a neighborhood of the critical point zero is described in the following proposition for the case where some eigenvalues of A have negative real parts and others have positive real parts.

Proposition 5.5 Consider the system

$$v' = Av + b(v),$$

where A is a constant matrix and $|b(v)| = o(|v|)$ as $v \to 0$. Assume that k eigenvalues of A have negative real parts and n-k have positive real parts. Then for any large t_o there exists in v-space a real k-dimensional manifold S containing the origin such that any solution v(t) of the above system with $v(t_o)$ on S satisfies $v(t) \to 0$ as $t \to \infty$. Moreover, there exists an $\eta > 0$ such that any solution v(t) near the origin but not on S at $t = t_o$ can not satisfy $|v(t)| \leq \eta$ for $t \geq t_o$.

Proof: See Coddington-Levinson [6], pages 330-333.

We end the discussion in this chapter by giving a proposition that is useful in showing that critical points of the system $v' = g(v)$ are stable in the large. The proposition makes use of the following definition:

Definition 5.4 A set D is called a set of monotonicity with respect to the system $v' = g(v)$ if and only if for each $k = 1,2,\ldots,n$ we have either $g_k(v) \geq 0$ or $g_k(v) \leq 0$ for all $v \in D$.

We note that if the solution $v(t)$ remains in a set of monotonicity for all $t \geq 0$, then each component of $v(t)$ is a monotone function of t.

Proposition 5.6 Let D_1, D_2 and D_3 be bounded subsets of R, and assume that each of them is a set of monotonicity with respect to the system $v' = g(v)$. Then

(i) If D_1 is invariant, then every solution of problem (P) with $v_o \in D_1$ approaches a critical point as $t \to \infty$.

(ii) If D_1, D_2, and $D_1 \cup D_2 \cup D_3$ are invariant, then every solution of problem (P) with $v_o \in D_1 \cup D_2 \cup D_3$ approaches a critical point as $t \to \infty$.

Proof:

(i) By Definition 5.1 the solution $v(t)$ of problem (P) with $v_o \in D_1$ exists for all $t \geq 0$, and $v(t) \in D_1$ for all $t \geq 0$.

The existence of the limit of the solution $v(t)$ as $t \to \infty$ follows from the monotonicity and boundedness of each of the components of the solution.

It now follows that the limit point is a critical point. Indeed, $v(t)$ goes monotonically toward its finite limit as $t \to \infty$. By continuity of g it follows that $v'(t)$ approaches a finite limit as $t \to \infty$. This finite limit must be equal to 0.

(ii) If $v_o \in D_1$ or $v_o \in D_2$, the conclusion follows from (i). Let $v_o \in D_3$. Then either the solution $v(t)$ remains in D_3 for all $t \geq 0$, or it does not. In the former case, the limit of the solution as $t \to \infty$ exists by boundedness and monotonicity of the components of the solution. Hence the limit point is a critical point. In the latter case there exists a finite $t_o > 0$ such that $v(t_o) \in D_1$ or $v(t_o) \in D_2$. The conclusion then follows from (i).

QED

Chapter 6

Model for Hermaphroditic Helminthiasis

6.1 Model Formulation

In this section we formulate a model for the transmission of hermaphroditic helminthiasis. We use the following notation:

N_1 = the total number of definitive hosts (assumed constant);

$\xi_k(t)$ = the number of mature parasites alive at time t inside definitive host number k, $k=1,2,\ldots,N_1$, $t \geq 0$;

$n_k = \xi_k(0)$ = the initial number of mature parasites alive inside definitive host number k, $k=1,2,\ldots,N_1$;

$n = \sum_{k=1}^{N_1} n_k$ = the total initial number of mature and alive parasites;

$R_{k;n}(t) = P\{\xi_k(t) = n\}$, $k=1,2,\ldots,N_1$, $n=0,1,2,\ldots$, $t \geq 0$;

N_2 = the total number of intermediate hosts (assumed constant);

$\eta(t)$ = the number of infected intermediate hosts at time $t \geq 0$;

$m_2 = N_2 y_o = \eta(0)$ = the initial number of infected intermediate hosts;

$\Pi_i(t) = P\{\eta(t) = i\}$, $i=0,1,2,\ldots,N_2$, $t \geq 0$.

We construct a model for the distributions of the $N_1 + 1$ random variables $\xi_1(t)$, $\xi_2(t),\ldots,$ $\xi_{N_1}(t)$, and $\eta(t)$. For each $k=1,2,\ldots,N_1$ we take $\xi_k(t)$ to be the number of individuals in a non-homogeneous immigration-death process with immigration rate $\nu_1 E\eta(t)$ and death rate μ_1 per individual. Furthermore, $\eta(t)$ is taken to be the number of individuals among N_2 in a non-homogeneous infection-recovery process with infection rate $\nu_2 E \sum_{k=1}^{N_1} \xi_k(t)$ per uninfected individual and recovery rate μ_2 per individual. The initial values of each of the $N_1 + 1$ random variables are assumed to be given. Using arguments in Chapters 3 and 4, we arrive at the following system of differential-difference equations with initial conditions for the unknown functions $R_{k;n}(t)$ and $\Pi_i(t)$:

$$R'_{k;n}(t) = \nu_1 \sum_{i=1}^{N_2} i\Pi_i(t)\, R_{k;n-1}(t) - \left[\nu_1 \sum_{i=1}^{N_2} i\Pi_i(t) + n\mu_1\right] R_{k;n}(t)$$

$$+ (n+1)\mu_1\, R_{k;n+1}(t),\ k=1,2,\ldots,N_1,\ n=0,1,2,\ldots,t \geq 0, \tag{6.1.1}$$

$$R_{k;n}(0) = \begin{cases} 1, & n = n_k, \\ 0, & n \neq n_k; \end{cases}$$

$$\Pi_i'(t) = \nu_2 \sum_{k=1}^{N_1} \sum_{n=1}^{\infty} n\, R_{k;n}(t)\,(N_2 - i + 1)\Pi_{i-1}(t)$$

$$- \left[\nu_2 \sum_{k=1}^{N_1} \sum_{n=1}^{\infty} n\, R_{k;n}(t)\,(N_2 - i) + i\mu_2\right]\Pi_i(t)$$

$$+ (i + 1)\mu_2\Pi_{i+1}(t),\ i=0,1,2,\ldots,N_2,\ t \geq 0, \qquad (6.1.2)$$

$$\Pi_i(0) = \begin{cases} 1, & i = m_2 = N_2 y_o, \\ 0, & i \neq m_2. \end{cases}$$

Here, $R_{k;-1}(t) = 0$, $k = 1,2,\ldots,N_1$, $t \geq 0$, and

$\Pi_{-1}(t) = \Pi_{N_2+1}(t) = 0$, $t \geq 0$.

Any non-negative functions $R_{k;n}(t)$ and $\Pi_i(t)$ that satisfy the above equations for all $t \geq 0$ and the above initial conditions and in addition the requirements that

$$\sum_{n=0}^{\infty} R_{k;n}(t) \leq 1,\ \sum_{n=1}^{\infty} n\, R_{k;n}(t) < \infty,\ k = 1,2,\ldots,N_1,\ t \geq 0,$$

$$\sum_{i=0}^{N_2} \Pi_i(t) = 1,\ t \geq 0,$$

are referred to as a solution of the model formulated above.

The parameters ν_1 and ν_2 can now be given the following interpretations: ν_1 is the rate of emission of cercariae by each infected intermediate host multiplied by the probability that a cercaria is successful in penetrating a given permanent host. Similarly, ν_2 is the rate of egg production of each mature parasite multiplied by the probability that an egg that has been laid leads to infection of a given non-infected intermediate host. We note also that $\nu_1 N_1$ is the rate of emission of successful cercariae by each infected intermediate host, and that $\nu_2 N_2$ would be the rate of emission of successful miracidia by each mature parasite, if we should recognize a miracidium as successful whenever it penetrates either a non-infected or an infected intermediate host.

For the further developments in this and succeeding chapters we introduce two so-called "transmission factors" T_1 and T_2 by the relations

$$T_1 = \frac{\nu_1 N_2}{\mu_1} \tag{6.1.3}$$

and

$$T_2 = \frac{\nu_2 N_1}{\mu_2} \, . \tag{6.1.4}$$

Heuristically, T_1 is a measure of transmission from intermediate host to definitive host, while T_2 is a measure of transmission

from definitive host to intermediate host. The transmission factors depend on the biological-environmental parameters of the community. In the next chapter we study their epidemiological importance, as for example their influence on the prevalence of the disease, on the proportion of infected intermediate hosts, and on a measure of public health.

The model proposed here has both stochastic and deterministic ingredients. The deterministic feature derives from the assumptions that immigration rates and infection rates are proportional to the expectations of $\eta(t)$ and $\sum_{k=1}^{N_1} \xi_k(t)$, respectively.

In the next section (6.2) we formulate and study an initial-value problem, derived from the model in this section. The results in Section 6.2 are used in Section 6.3 to establish existence and uniqueness of the solution of the model. They are also used in the study of the epidemiology of hermaphroditic helminthiasis in Chapter 7.

6.2 An Initial-Value Problem

For the discussion in this section we assume that a solution exists of the model formulated in Section 6.1. We define the four functions $Y(t)$, $y(t)$, $X(t)$ and $x(t)$ by the following relations:

$$Y(t) = N_2 y(t) = \sum_{i=1}^{N_2} i\Pi_i(t) \tag{6.2.1}$$

and

$$X(t) = N_1 x(t) = \sum_{k=1}^{N_1} \sum_{n=1}^{\infty} n\, R_{k;n}(t). \tag{6.2.2}$$

For any $k=1,2,\ldots,N_1$, consider now the system of differential-difference equations with initial data given in (6.1.1). Since the functions $\Pi_i(t)$ appearing in (6.2.1) are solutions of the model in Section 6.1, they are all non-negative, bounded, and continuous. The function $Y(t)$ defined in (6.2.1) has therefore the same three properties. The systems (6.1.1) and (3.1.9) are now seen to be formally identical. In Chapter 3 we showed that system (3.1.9) has a unique solution, and we exhibited, in (3.2.4), an expression for the expectation of the random variable $\xi(t)$ whose distribution is given by the functions $R_n(t)$. We therefore conclude

$$\sum_{n=1}^{\infty} n\, R_{k;n}(t) = \beta(t) + n_k \alpha(t), \tag{6.2.3}$$

where the functions α and β are defined in (3.2.1) and (3.2.2), respectively. By inserting expression (6.2.3) into (6.2.2) we find

$$x(t) = \beta(t) + \frac{n}{N_1}\,\alpha(t). \tag{6.2.4}$$

By differentiation of this expression we find that the function $x(t)$ satisfies the differential equation

$$x'(t) = -\mu_1 x(t) + T_1 \mu_1 y(t). \tag{6.2.5}$$

Its initial value at $t = 0$ is denoted by x_o, where

$$x_o = x(0) = \frac{n}{N_1}\,. \tag{6.2.6}$$

Now consider the system of differential-difference equations with initial data given in (6.1.2). From relation (6.2.4) follows that the function $x(t)$, and hence also the function $X(t)$, is non-negative, bounded, and continuous for $t \geq 0$. The systems (6.1.2) and (4.1.1) are now seen to be identical. In Chapter 4 we showed that system (4.1.1) has a unique solution, and we found, in (4.2.5) and (4.2.6), an initial-value problem that is satisfied by the expectation of the random variable whose distribution is given by the function $\Pi_i(t)$. Hence the function $Y(t)$ satisfies the same initial-value problem. We find from (4.2.5) and (4.2.6) that

$$y'(t) = T_2\mu_2 x(t)(1-y(t))-\mu_2 y(t) \tag{6.2.7}$$

and

$$y(0) = y_o. \tag{6.2.8}$$

We now introduce the notation

$$u(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

$$f(u) = \begin{pmatrix} f_1(u) \\ f_2(u) \end{pmatrix} = \begin{pmatrix} -\mu_1 x+T_1\mu_1 y \\ T_2\mu_2 x(1-y)-\mu_2 y \end{pmatrix},$$

$$u_o = \begin{pmatrix} x_o \\ y_o \end{pmatrix}.$$

The initial-value problem (6.2.5)-(6.2.8) can then be written in the form

$$\begin{cases} u' = f(u), \\ u(0) = u_o. \end{cases}$$

This initial-value problem is called Problem (6.2). We devote the following four subsections (6.2.1-6.2.4) to an investigation of properties of the solution of Problem (6.2).

6.2.1 Existence, Uniqueness, and Boundedness of the Solution

In this subsection we prove a theorem concerning global existence (i.e. existence for all $t \geq 0$), uniqueness, and boundedness of the solution of Problem (6.2). We let x_1 denote any number satisfying the inequality $x_1 \geq T_1$, and we define a set B by putting

$$B = \{(x,y): \ 0 \leq x \leq x_1, \ 0 \leq y \leq 1\}. \qquad (6.2.1.1)$$

Lemma 6.2.1.1

The set B is invariant with respect to the system $u' = f(u)$ of Section 6.2.

Proof:

By Proposition 5.2 it suffices to show that every point on ∂B is an ingress point of B. We consider several subsets of the boundary ∂B. In each case we use $\bar{u} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$ to denote a point belonging to the subset of the boundary under consideration. We then consider the problem

$$\begin{cases} u' = f(u), \\ u(0) = \bar{u}. \end{cases}$$

Since the Jacobian matrix $J = \frac{\partial f}{\partial u}$ is continuous, this problem has a unique solution in some time-interval $-\varepsilon < t < \varepsilon$, $\varepsilon > 0$. We denote the solution by $u(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$. We then have a tangent to the solution given by the vector $T(t) = (x'(t), y'(t))$, and $T(0) = (f_1(\bar{u}), f_2(\bar{u}))$. We use N to denote a normal to ∂B directed into B.

Consider now the following four lines:

1) $\bar{x} = 0$, $0 < \bar{y}$.

   Here, $N = (1, 0)$ and

   $T(0) = (T_1\mu_1\bar{y}, f_2(\bar{u}))$.

   Hence $N \cdot T(0) = T_1\mu_1\bar{y} > 0$.

2) $\bar{x} = x_1$, $\bar{y} < 1$.

   Here, $N = (-1, 0)$ and

   $T(0) = (-\mu_1 x_1 + T_1\mu_1\bar{y}, f_2(\bar{u}))$.

   Hence $N \cdot T(0) = \mu_1 x_1 - T_1\mu_1\bar{y} > 0$.

3) $0 < \bar{x}$, $\bar{y} = 0$.

   Here, $N = (0, 1)$ and

   $T(0) = (f_1(\bar{u}), T_2\mu_2\bar{x})$.

   Hence $N \cdot T(0) = T_2\mu_2\bar{x} > 0$.

4) $\bar{y} = 1$.

Here, $N = (0, -1)$ and

$T(0) = (f_1(\bar{u}), -\mu_2)$.

Hence $N \cdot T(0) = \mu_2 > 0$.

From these results and Proposition 5.3 we conclude that all points on ∂B, except possibly $(0,0)$, $(0,1)$, $(x_1,0)$, and $(x_1,1)$, are ingress points of $\overset{\circ}{B}$. By also applying Proposition 5.1 (ii) we find $(0,1)$ and $(x_1,0)$ to be ingress points of $\overset{\circ}{B}$.

In case $x_1 > T_1$, the result in 2) applies also for $\bar{y} = 1$. In case $x_1 = T_1$ we find for the point $\bar{x} = x_1$, $\bar{y} = 1$ that

$$x'(0) = -\mu_1 T_1 + T_1 \mu_1 = 0,$$

$$y'(0) = -\mu_2 < 0,$$

$$x''(0) = -T_1 \mu_1 \mu_2 < 0.$$

Hence the function $x(t)$ has a local maximum at $t = 0$. We conclude that $(x_1, 1)$ is in either case an ingress point of the set $\overset{\circ}{B}$.

For the point $\bar{x} = \bar{y} = 0$, the solution is

$$u(t) = \begin{pmatrix} 0 \\ \\ 0 \end{pmatrix}, \quad t \geq 0.$$

Hence $u(t) \in \partial B$, $t \geq 0$, which shows that $(0, 0)$ is an ingress point of the closed set B. QED

The main result of this subsection is given in the following theorem.

Theorem 6.2.1.1

Problem (6.2) has a unique solution $u(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$. The solution exists for all $t \geq 0$. The solution components have the following bounds: $0 \leq x(t) \leq x_o \vee T_1$, $0 \leq y(t) \leq 1$, $t \geq 0$.

Proof:

Define $x_1 = x_o \vee T_1$ and let the rectangle B be defined as in (6.2.1.1). We then have $x_1 \geq T_1$. From Lemma 6.2.1.1 we conclude that the set B is invariant with respect to the system $u' = f(u)$. The conclusion of the theorem now follows from Definition 5.1, since the initial point u_o belongs to the set B. QED

6.2.2 The Critical Points

In this subsection we study the critical points of Problem (6.2). A point is said to be a critical point of this problem if it satisfies two criteria:

1) It is a critical point of the system $u' = f(u)$ of Section 6.2.
2) It belongs to the invariant set B defined in (6.2.1.1) with $x_1 = x_o \vee T_1$.

Thus, no critical point of the system $u' = f(u)$ which falls outside the set B is regarded as a critical point of the problem. By Theorem 6.2.1.1 no such point can be a solution or a limit point of a solution of Problem (6.2).

We introduce the notation

$$\overset{\circ}{x} = T_1 - \frac{1}{T_2}, \tag{6.2.2.1}$$

$$\overset{\circ}{y} = 1 - \frac{1}{T_1 T_2}, \tag{6.2.2.2}$$

$$\overset{\circ}{u} = \begin{pmatrix} \overset{\circ}{x} \\ \overset{\circ}{y} \end{pmatrix}, \tag{6.2.2.3}$$

$$0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{6.2.2.4}$$

The following lemma summarizes the results:

Lemma 6.2.2.1

(i) Assume $T_1 T_2 \leq 1$. Then Problem (6.2) has 0 as its only critical point.

(ii) Assume $T_1 T_2 > 1$. Then Problem (6.2) has two distinct critical points: 0 and $\overset{\circ}{u}$.

Proof: The critical points of the system $u' = f(u)$ are solutions to the system of equations $f(u) = 0$, or

$$\begin{cases} y = \frac{1}{T_1} x, \\ y = \frac{T_2 x}{1 + T_2 x}. \end{cases}$$

This system of equations has two solutions, i.e.

$$\begin{cases} x = 0, \\ y = 0, \end{cases}$$

and

$$\begin{cases} x = \overset{\circ}{x}, \\ y = \overset{\circ}{y}. \end{cases}$$

The first of these points belongs to the set B for all values of T_1 and T_2. The second of these points belongs to the set B for $T_1T_2 \geq 1$. QED

We note that even though the system $u' = f(u)$ contains four parameters, i.e. μ_1, μ_2, T_1, and T_2, the number of critical points and their coordinates depend on only two of these parameters, i.e. T_1 and T_2.

6.2.3 Local Stability of the Critical Points

Local stability properties of the critical points of Problem (6.2) are studied in this subsection with the aid of Proposition 5.4.

Theorem 6.2.3.1

(i) Assume $T_1T_2 < 1$. Then the critical point 0 is asymptotically stable.

(ii) Assume $T_1T_2 > 1$. Then the critical point 0 is unstable, while the critical point $\overset{\circ}{u}$ is asymptotically stable.

Proof: We write the system $u' = f(u)$ in the form

$$u' = Au + b(u),$$

where

$$A = \begin{pmatrix} -\mu_1 & T_1\mu_1 \\ T_2\mu_2 & -\mu_2 \end{pmatrix}$$

and

$$b(u) = \begin{pmatrix} 0 \\ -T_2\mu_2 xy \end{pmatrix}.$$

Hence $|b(u)| = o(|u|)$ as $u \to 0$, so Proposition 5.4 applies. The eigenvalues of A are

$$-\frac{\mu_1+\mu_2}{2} \pm \frac{\sqrt{(\mu_1-\mu_2)^2+4T_1T_2\mu_1\mu_2}}{2}$$

The two eigenvalues of A are therefore real and distinct. The smaller one is negative while the larger one has the same sign as $T_1T_2 - 1$. We conclude that the critical point 0 is asymptotically stable for $T_1T_2 < 1$, and it is unstable for $T_1T_2 > 1$.

In order to investigate the stability of the critical point $\overset{\circ}{u}$ (assuming $T_1T_2 \geq 1$) we introduce

$$\tilde{x} = x - \overset{\circ}{x},$$

$$\tilde{y} = y - \overset{\circ}{y},$$

and

$$\tilde{u} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}.$$

From the system $u' = f(u)$ we derive

$$\tilde{u}' = \tilde{A}\,\tilde{u} + \tilde{b}(\tilde{u}),$$

where

$$\tilde{A} = \begin{pmatrix} -\mu_1 & T_1\mu_1 \\ \dfrac{\mu_2}{T_1} & -T_1T_2\mu_2 \end{pmatrix}$$

and

$$\tilde{b}(\tilde{u}) = \begin{pmatrix} 0 \\ -T_2\mu_2\tilde{x}\tilde{y} \end{pmatrix}.$$

Again $|\tilde{b}(\tilde{u})| = o(|\tilde{u}|)$ as $\tilde{u} \to 0$, so Proposition 5.4 applies. The eigenvalues of $\tilde{A}$ are

$$-\frac{\mu_1+T_1T_2\mu_2}{2} \pm \frac{\sqrt{(\mu_1-T_1T_2\mu_2)^2+4\mu_1\mu_2}}{2}.$$

The two eigenvalues of $\tilde{A}$ are therefore real and distinct. The smaller one is negative while the larger one has the same sign as $1 - T_1T_2$. We conclude that the critical point $\overset{\circ}{u}$ is asymptotically stable for $T_1T_2 > 1$. QED

6.2.4 Stability in the Large of the Critical Points

In this subsection we prove some stability properties in the large of the critical points of Problem (6.2). In addition to notation that has been used earlier in this chapter, we introduce the following:

$$X^{\circ} = \{(x,y): \quad y = \frac{1}{T_1} x\}, \tag{6.2.4.1}$$

$$Y^{\circ} = \left\{(x,y): \quad y = \frac{T_2 x}{1 + T_2 x}\right\}, \tag{6.2.4.2}$$

$$X^{+} = \{(x,y): \quad y > \frac{1}{T_1} x\},$$

$$X^{-} = \{(x,y): \quad y < \frac{1}{T_1} x\},$$

$$Y^{+} = \left\{(x,y): \quad y < \frac{T_2 x}{1 + T_2 x}\right\},$$

$$Y^{-} = \left\{(x,y): \quad y > \frac{T_2 x}{1 + T_2 x}\right\},$$

$$R_1 = X^{+} \cap Y^{+},$$
$$R_2 = X^{-} \cap Y^{-},$$
$$R_3 = X^{+} \cap Y^{-},$$
$$R_4 = X^{-} \cap Y^{+}.$$

We use the rectangle B of (6.2.1.1) with $x_1 = x_o \vee T_1$ and put $S_i = R_i \cap B$, $i = 1, \ldots, 4$. Figures 6.2.4.1 and 6.2.4.2 show the configurations of the above sets in cases $T_1T_2 \leq 1$ and $T_1T_2 > 1$, respectively. Using the above notation we now establish the following lemma.

Lemma 6.2.4.1

The sets $\bar{S}_1$, $\bar{S}_2$, $\bar{S}_1 \cup \bar{S}_2 \cup \bar{S}_3$, and $\bar{S}_1 \cup \bar{S}_2 \cup \bar{S}_4$ are invariant with respect to the system $u' = f(u)$ of Section 6.2.

Proof:

The four sets mentioned in the lemma are all bounded and closed. It suffices therefore, by Proposition 5.2, to show that the boundary points of each set are ingress points of the corresponding set. But from Lemma 6.2.1.1 we find that all boundary points of B are ingress points of B. By an application of Proposition 5.1 (iii) we therefore conclude that it suffices to show that the boundary points of each of the sets $\bar{R}_1$, $\bar{R}_2$, $\bar{R}_1 \cup \bar{R}_2 \cup \bar{R}_3$, and $\bar{R}_1 \cup \bar{R}_2 \cup \bar{R}_4$ are ingress points of the corresponding set. Now every boundary point of each of these sets belongs to $X^o \cup Y^o$. Thus we consider the initial-value problem

$$\begin{cases} u' = f(u), \\ u(0) = \bar{u}, \end{cases} \qquad \text{(I)}$$

with $\bar{u} \in X^o \cup Y^o$. By allowing $\bar{u}$ to range over the whole set $X^o \cup Y^o$, all boundary points of the four sets

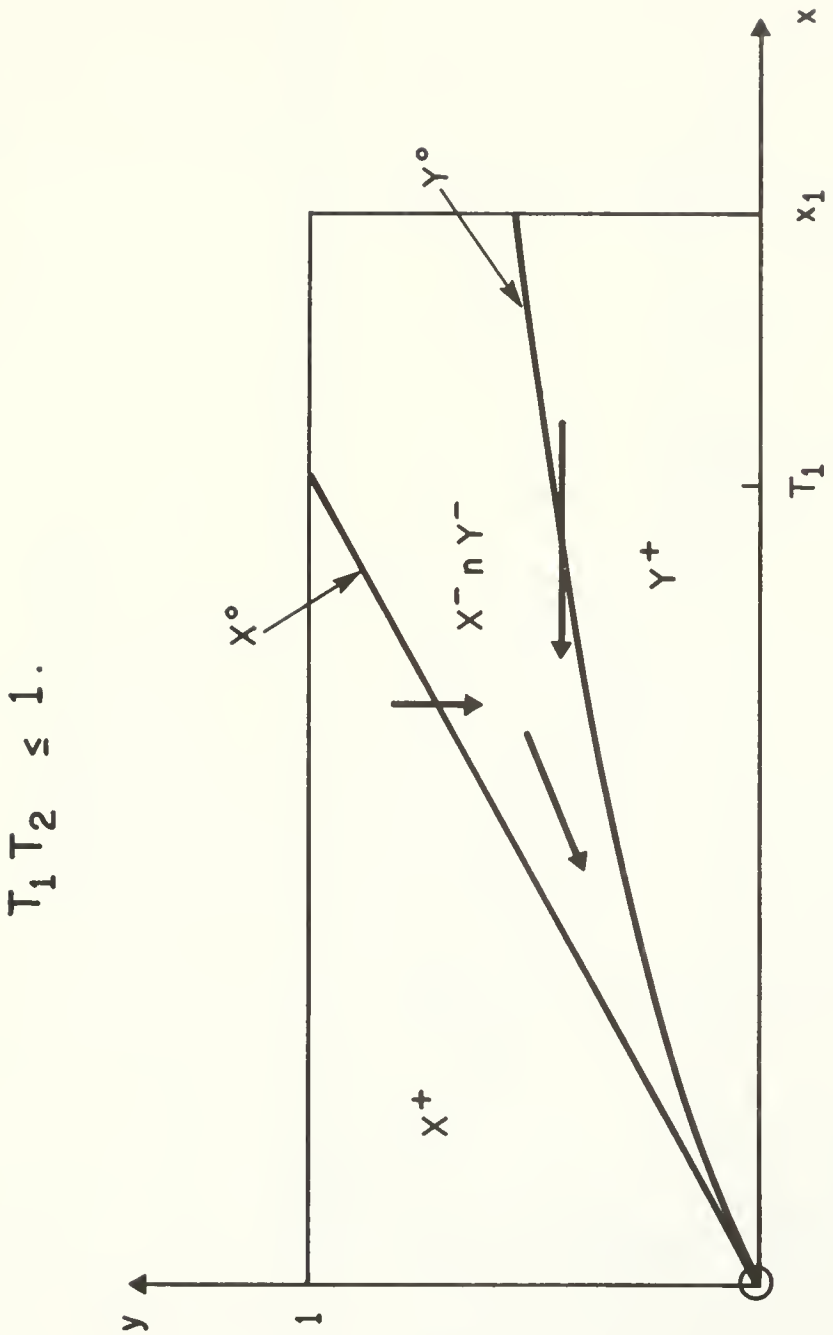

FIGURE 6.2.4.1

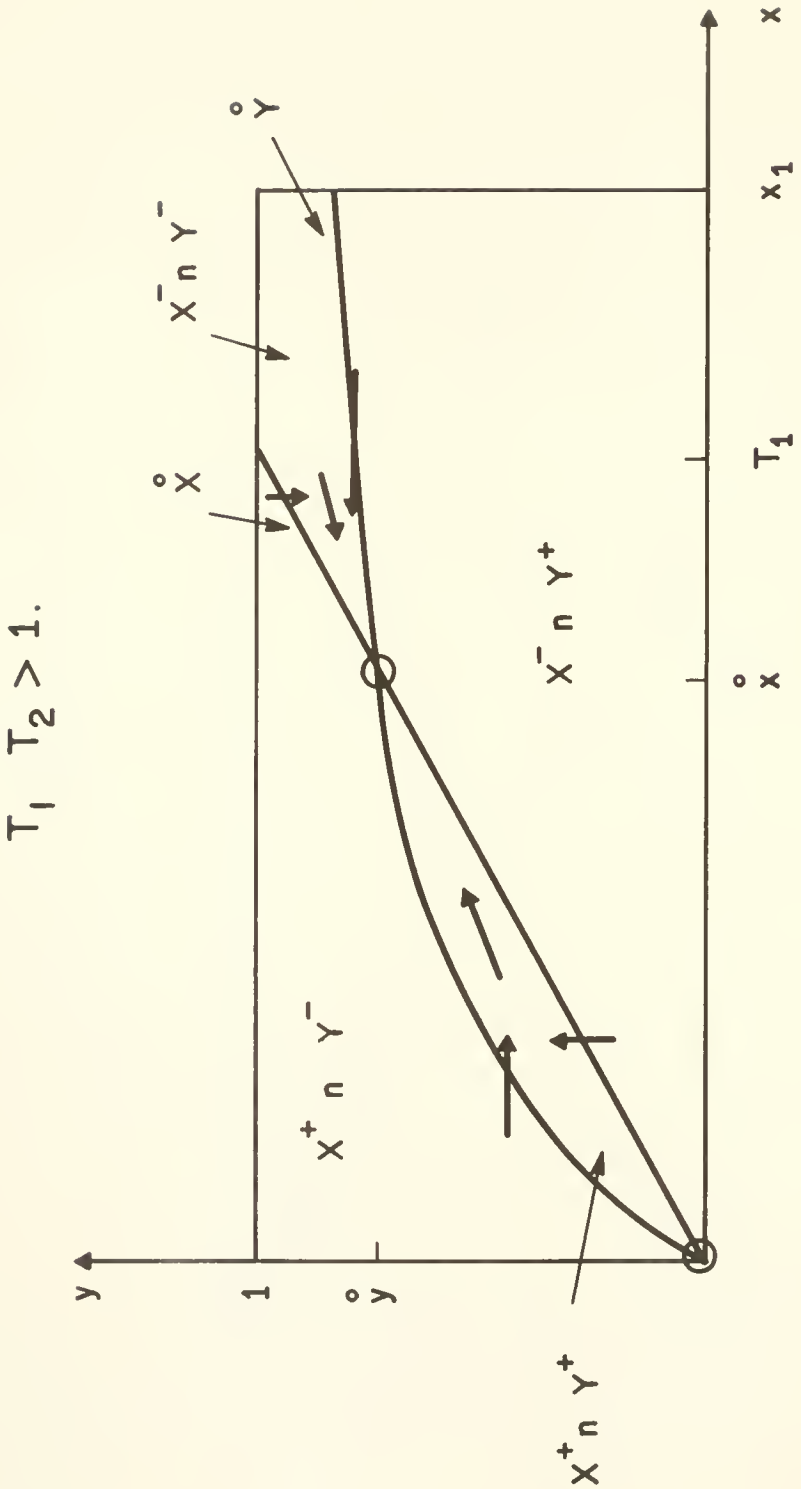

FIGURE 6.2.4.2

$\bar{R}_1$, $\bar{R}_2$, $\bar{R}_1 \cup \bar{R}_2 \cup \bar{R}_3$, and $\bar{R}_1 \cup \bar{R}_2 \cup \bar{R}_4$ are covered. Now the set $X^\circ \cup Y^\circ$ can be partitioned into the union of five disjoint sets as follows:

$$X^\circ \cup Y^\circ = \bigcup_{i=1}^{5} A_i,$$

where

$$A_1 = X^\circ \cap Y^+,$$
$$A_2 = X^\circ \cap Y^-,$$
$$A_3 = X^+ \cap Y^\circ,$$
$$A_4 = X^- \cap Y^\circ,$$

and

$$A_5 = X^\circ \cap Y^\circ.$$

Consider now $\bar{u} \in A_1$. Then $\bar{u}$ belongs to the boundaries of those portions of the sets X^+ and X^- that are contained in Y^+, i.e. $\bar{u}$ belongs to the boundaries of the sets $R_1 = X^+ \cap Y^+$ and $R_4 = X^- \cap Y^+$. We note also that $\bar{u}$ does not belong to the boundaries of R_2 or R_3, since $\bar{u} \in Y^+$. In a similar way we find: If $\bar{u} \in A_2$, then $\bar{u}$ is a boundary point of R_2 and R_3 but not of R_1 or R_4. If $\bar{u} \in A_3$, then $\bar{u}$ is a boundary point of R_1 and R_3 but not of R_2 or R_4. If $\bar{u} \in A_4$, then $\bar{u}$ is a boundary point of R_2 and R_4 but not of R_1 or R_3.

In proving the lemma, we are concerned with boundary points of the sets $\bar{R}_1$, $\bar{R}_2$, $\bar{R}_1 \cup \bar{R}_2 \cup \bar{R}_3$, and $\bar{R}_1 \cup \bar{R}_2 \cup \bar{R}_4$. From the preceding paragraph we find: If $\bar{u} \in A_1$, then it suffices to show that $\bar{u}$ is an ingress point of $\bar{R}_1$. If $\bar{u} \in A_2$, then it suffices to show that $\bar{u}$ is an ingress point of $\bar{R}_2$. If $\bar{u} \in A_3$, then it suffices to show that $\bar{u}$ is an ingress point or $\bar{R}_1$. If $\bar{u} \in A_4$, then it suffices to show that $\bar{u}$ is an ingress point of $\bar{R}_2$.

If $\bar{u} \in A_5$, then $\bar{u}$ is a critical point. Hence $\bar{u}$ is an ingress point of the closure of each set of which it is a boundary point.

Now define a function $\bar{X}$ by

$$\bar{X}(x,y) = y - \frac{x}{T_1} .$$

Then X° is the straight line defined by $\bar{X}(x,y) = 0$. The vector

$$N_X = (- \frac{1}{T_1} , 1)$$

is normal to X° and is directed into the set X^+. Problem (I) with $\bar{u} \in X^\circ$ has a solution with tangent at $\bar{u}$ given by the vector

$$T(0) = (0, f_2(\bar{u})).$$

Hence $N_X \cdot T(0) = f_2(\bar{u})$. Now $f_2(\bar{u}) > 0$ for $\bar{u} \in Y^+$ and $f_2(\bar{u}) < 0$ for $\bar{u} \in Y^-$.

With $\bar{u} \in A_1 = X^\circ \cap Y^+$ we find by Proposition 5.3 that $\bar{u}$ is an ingress point of X^+. Proposition 5.1 (i) then shows that $\bar{u}$ is an ingress point of $R_1 = X^+ \cap Y^+$.

With $\bar{u} \in A_2 = X^\circ \cap Y^-$ we find by the observation following Proposition 5.3 that $\bar{u}$ is an ingress point of X^-. Proposition 5.1 (i) then shows that $\bar{u}$ is an ingress point of $R_2 = X^- \cap Y^-$.

Next define a function $\bar{Y}$ by

$$\bar{Y}(x,y) = \frac{T_2 x}{1 + T_2 x} - y.$$

Then Y° is the curve defined by $\bar{Y}(x,y) = 0$. The vector

$$N_Y = \left(\frac{T_2}{(1+T_2 x)^2}, -1 \right)$$

is normal to Y° and is directed into the set Y^+. Problem (I) with $\bar{u} \in Y^\circ$ has a solution with

$$T(0) = (f_1(\bar{u}), 0).$$

Hence

$$N_Y \cdot T(0) = \frac{T_2}{(1+T_2 x)^2} f_1(\bar{u}).$$

This inner product is positive for $\bar{u} \in X^+$ and negative for $\bar{u} \in X^-$.

With $\bar{u} \in A_3 = X^+ \cap Y^\circ$ we find by Proposition 5.3 that $\bar{u}$ is an ingress point of Y^+. Proposition 5.1 (i) then shows that $\bar{u}$ is an ingress point of $R_1 = X^+ \cap Y^+$.

With $\bar{u} \in A_4 = X^- \cap Y^\circ$ we find by the observation following Proposition 5.3 that $\bar{u}$ is an ingress point of Y^-. Proposition 5.1 (i) then shows that $\bar{u}$ is an ingress point of $R_2 = X^- \cap Y^-$. QED

We now proceed to the main result of this subsection.

Theorem 6.2.4.1 Consider the initial-value problem

$$\begin{cases} u' = f(u) \\ u(0) = u_o \end{cases}$$

of Section 6.2.

I. If $u_o = 0$, the solution is $u(t) = 0$, $t \geq 0$.

II. If $T_1 T_2 \leq 1$, the solution $u(t)$ approaches 0 as $t \to \infty$.

III. If $u_o \neq 0$ and $T_1 T_2 > 1$, the solution $u(t)$ approaches $\overset{\circ}{u}$ as $t \to \infty$, where $\overset{\circ}{u}$ is given in (6.2.2.3).

Proof:

We show first that every solution approaches a critical point as $t \to \infty$.

From the definitions of the sets $S_1, \ldots, S_4$ it follows that these sets, as well as their closures, are sets of monotonicity with respect to the system $u' = f(u)$. Lemma 6.2.4.1 shows that the sets $\overline{S}_1$, $\overline{S}_2$, $\overline{S}_1 \cup \overline{S}_2 \cup \overline{S}_3$, and $\overline{S}_1 \cup \overline{S}_2 \cup \overline{S}_4$ are invariant. We now apply Proposition 5.6 (ii) twice. In the first application we take $D_1 = \overline{S}_1$, $D_2 = \overline{S}_2$, and $\overline{D}_3 = \overline{S}_1 \cup \overline{S}_2 \cup \overline{S}_3$. In the second application we take $D_1 = \overline{S}_1$, $D_2 = \overline{S}_2$, and $D_3 = \overline{S}_1 \cup \overline{S}_2 \cup \overline{S}_4$. We conclude that every solution of Problem (6.2) approaches a critical point as $t \to \infty$.

Statement I of the theorem follows since $u_o = 0$ is a critical point.

Statement II of the theorem follows since 0 is in this case the only critical point of the problem, as shown in Lemma 6.2.2.1.

In order to complete the proof of the theorem, it suffices to show that if $u_o \neq 0$ and $T_1T_2 > 1$, then the solution approaches $\overset{o}{u}$ as $t \to \infty$. This is done by considering the exhaustive cases $u_o \in \bar{S}_1$, $u_o \in \bar{S}_2$, $u_o \in S_3$ and $u_o \in S_4$.

If $u_o \in \bar{S}_1$, then x_o is strictly positive since $u_o \neq 0$. The solution remains in $\bar{S}_1$ for all $t \geq 0$ since $\bar{S}_1$ is invariant. Now the x-component of the solution is non-decreasing in $\bar{S}_1$ since $x'(t) = f_1(u) \geq 0$ in $\overline{X^+}$. Since the solution approaches a critical point as $t \to \infty$, and since the limit of the solution must have strictly positive x-coordinate, we conclude that the solution approaches $\overset{o}{u}$ as $t \to \infty$.

If $u_o \in \bar{S}_2$, then the solution remains in $\bar{S}_2$ for all $t \geq 0$ since $\bar{S}_2$ is invariant. But we also have that the solution approaches a critical point as $t \to \infty$. The only critical point belonging to $\bar{S}_2$ is $\overset{o}{u}$. Hence we conclude that the solution approaches $\overset{o}{u}$ as $t \to \infty$.

If $u_o \in S_3$, then either the solution remains in S_3 for all $t \geq 0$, or it does not. In the former case the x-component of the solution is a strictly increasing function of t, since $x'(t) = f_1(u) > 0$ in X^+. Since the solution approaches a critical point as $t \to \infty$, and since the limit of the solution must have strictly positive x-coordinate, we conclude in this case that the solution approaches $\overset{\circ}{u}$ as $t \to \infty$. In the latter case above, the solution enters one of the sets $\overline{S}_1$ or $\overline{S}_2$ at some finite time $t_o > 0$, since $\overline{S}_1 \cup \overline{S}_2 \cup S_3 = \overline{S}_1 \cup \overline{S}_2 \cup \overline{S}_3$ is invariant. If the solution enters $\overline{S}_1$, the entrance point is not the critical point 0. The reason is that a solution of an autonomous system of differential equations can never reach any of its critical points in finite time unless a critical point serves as initial value. From the argument above for $u_o \in \overline{S}_1$ we conclude that the solution approaches $\overset{\circ}{u}$ as $t \to \infty$. If the solution enters $\overline{S}_2$ at time $t_o > 0$, we use the argument above for $u_o \in \overline{S}_2$ to conclude that the solution approaches $\overset{\circ}{u}$ as $t \to \infty$.

If $u_o \in S_4$, then an argument very similar to the one just given for $u_o \in S_3$ shows that the solution approaches $\overset{\circ}{u}$ as $t \to \infty$. QED

6.3 Existence and Uniqueness of the Solution of the Model

Theorem 6.3.1

There exists a unique solution of the model of Section 6.1.

Proof:

In order to prove uniqueness, assume that there exists at least one solution of the model of Section 6.1. From the arguments in Section 6.2 it then follows that the functions $x(t)$ and $y(t)$ defined by (6.2.2) and (6.2.1), respectively, satisfy the initial-value problem of Section 6.2. From Theorem 6.2.1.1 we conclude that these two functions are uniquely determined, and that they are non-negative, bounded, and continuous. For each $k=1,2,\ldots,N_1$, consider the system (6.1.1) of differential-difference equations with initial values for the functions $R_{k;n}(t)$. In this system, replace the expression $\sum_{i=1}^{N_2} i\Pi_i(t)$ by $N_2 y(t) = Y(t)$. The resulting system is formally identical with system (3.1.9). But the latter system has a unique solution as noted at the end of Section 3.1. Hence the functions $R_{k;n}(t)$ are uniquely determined. Furthermore, consider the system (6.1.2) of differential-difference equations with initial values for the functions $\Pi_i(t)$. In this system, replace the expression $\sum_{k=1}^{N_1} \sum_{n=1}^{\infty} n R_{k;n}(t)$ by $N_1 x(t) = X(t)$. The resulting system is formally identical with system (4.1.1). But the latter system has a unique solution as noted at the end of Section 4.1. Hence the functions $\Pi_i(t)$ are uniquely determined. This concludes the proof of the uniqueness statement.

In order to prove existence of a solution to the model, let $\begin{pmatrix} x^*(t) \\ y^*(t) \end{pmatrix}$, $t \geq 0$, denote the solution of Problem (6.2). Both of the functions $x^*(t)$ and $y^*(t)$ are then non-negative, continuous, and bounded by Theorem 6.2.1.1. Put $X^*(t) = N_1 x^*(t)$ and $Y^*(t) = N_2 y^*(t)$. For each $k = 1,2,\ldots,N_1$, consider the system (3.1.9) with n_o replaced by n_k, $R_n(t)$ replaced by $R_{k;n}(t)$, and with the above function $Y^*(t)$. From Chapter 3 it follows that this system can be solved uniquely for the functions $R_{k;n}(t)$. Consider also the system (4.1.1) with the above function $X^*(t)$. From Chapter 4 it follows that this system can be solved uniquely for the functions $\Pi_i(t)$. We claim that the resulting collection of functions $R_{k;n}(t)$, $k = 1,2,\ldots,N_1$, $n = 1,2,\ldots$, and $\Pi_i(t)$, $i = 1,2,\ldots,N_2$, constitutes a solution of the model of Section 6.1. In order to prove this claim we compare systems (3.1.9) and (6.1.1) and also systems (4.1.1) and (6.1.2). We find from this comparison that is suffices to show that

$$x^*(t) = \frac{1}{N_1} \sum_{k=1}^{N_1} \sum_{n=1}^{\infty} n\, R_{k;n}(t) \tag{6.3.1}$$

and

$$y^*(t) = \frac{1}{N_2} \sum_{i=1}^{N_2} i\Pi_i(t). \tag{6.3.2}$$

The expression $\sum_{n=1}^{\infty} n\, R_{k;n}(t)$ is now the expectation of the random variable $\xi_k(t)$ whose distribution is given by

$$R_{k;n}(t) = P\{\xi_k(t) = n\}, \qquad n=0,1,2,\ldots .$$

From expression (3.2.4) we find

$$\sum_{n=1}^{\infty} n\, R_{k;n}(t) = \beta(t) + n_k\alpha(t),$$

where the functions α and β are defined in (3.2.1) and (3.2.2), respectively, and $Y(s) = N_2 y^*(s)$ in (3.2.2). Hence

$$\frac{1}{N_1} \sum_{k=1}^{N_1} \sum_{n=1}^{\infty} n\, R_{k;n}(t) = \beta(t) + \frac{n}{N_1}\alpha(t).$$

From this relation we derive the differential equation

$$\frac{d}{dt}\left[\frac{1}{N_1} \sum_{k=1}^{N_1} \sum_{n=1}^{\infty} n\, R_{k;n}(t)\right] = -\mu_1 \frac{1}{N_1} \sum_{k=1}^{N_1} \sum_{n=1}^{\infty} n\, R_{k;n}(t) + T_1\mu_1 y^*(t).$$

At $t=0$ we have

$$\frac{1}{N_1} \sum_{k=1}^{N_1} \sum_{n=1}^{\infty} n\, R_{k;n}(0) = \frac{n}{N_1} = x_o.$$

A comparison with (6.2.5) and (6.2.6) shows that $x^*(t)$ and $\frac{1}{N_1} \sum_{k=1}^{N_1} \sum_{n=1}^{\infty} n\, R_{k;n}(t)$ satisfy the same linear initial-value problem. This problem has a unique solution; hence we conclude that (6.3.1) holds.

The expression $\sum_{i=1}^{N_2} i\Pi_i(t)$ in (6.3.2) is the expectation of the random variable $\eta(t)$ whose distribution is given by

$$\Pi_i(t) = P\{\eta(t) = i\}, \qquad i=0,1,\ldots,N_2.$$

From (4.2.5) we find the following differential equation to be satisfied:

$$\frac{d}{dt}\left[\frac{1}{N_2}\sum_{i=1}^{N_2} i\Pi_i(t)\right] = T_2\mu_2 x^*(t)\left[1-\frac{1}{N_2}\sum_{i=1}^{N_2} i\Pi_i(t)\right]-\mu_2\frac{1}{N_2}\sum_{i=1}^{N_2} i\Pi_i(t).$$

At t=0 we have from (4.2.6)

$$\frac{1}{N_2}\sum_{i=1}^{N_2} i\Pi_i(0) = y_o.$$

A comparison with (6.2.7) and (6.2.8) now shows that $y^*(t)$ and $\frac{1}{N_2}\sum_{i=1}^{N_2} i\Pi_i(t)$ satisfy the same linear initial-value problem. We conclude that (6.3.2) holds, since the initial-value problem has a unique solution. QED

Chapter 7

Epidemiology of Hermaphroditic Helminthiasis

7.1 The Threshold

In Theorem 6.2.4.1 we proved that every solution of Problem (6.2) approaches the critical point 0 if $T_1T_2 \leq 1$, and that every solution (with one exception) approaches the non-vanishing critical point $\overset{\circ}{u}$ if $T_1T_2 > 1$. We refer to this phenomenon as a threshold effect. In this section we give epidemiological interpretations of this threshold effect.

In any given community we assume that the biological-environmental parameters that determine the transmission factors are fixed, although they may change under the influence of actions taken by man.

Consider now a community where no disease exists, and assume first that $T_1T_2 \leq 1$. An introduction into such a community of any amount of disease through infected definitive or intermediate hosts will not lead to the establishment of a positive prevalence of disease in the community. On the other hand, if $T_1T_2 > 1$, then the introduction of even a small number of infected definitive or intermediate hosts will lead to a spread of the disease, and the prevalence of the disease will approach a constant positive level. In balance with this level of prevalence

there will be a proportion of infected intermediate hosts, and it will correspond to a constant probability of infection for each definitive host in a short time interval of length Δt. We shall introduce a so-called public health factor as a measure of this probability of infection. In Sections 7.2-7.4 we study the dependence of prevalence, proportion of infected intermediate hosts, and public health factor on biological-environmental parameters.

The threshold effect described above is a familiar one in mathematical epidemiology. A similar qualitative phenomenon was described in 1927 by Kermack-McKendrick [12].

The existence of this threshold phenomenon can be used to discuss eradication of the disease when it is established near the positive steady-state level of endemicity. It is seen that in order to eradicate the disease it is necessary and sufficient to reduce the product of the transmission factors T_1 and T_2 to a value less than or equal to 1. The proportionate reduction in T_1 that is required in order to achieve eradication for fixed T_2 is therefore equal to the proportionate reduction in T_2 that is required in order to achieve eradication for fixed T_1. We express this fact by saying that for eradication purposes a reduction in T_1 is as effective as a reduction in T_2.

7.2 The Prevalence

In this section we study the prevalence of the disease in the case where $T_1T_2 > 1$ and $u_o \neq 0$. The

prevalence is measured by the proportion of definitive hosts that contain at least one parasite in their bodies, i.e. by $P\{\xi(t) > 0\}$. We assume that a sufficiently long time has elapsed so that the prevalence is close to its asymptotic or steady-state value. From (3.5.3) we find that $P\{\xi(t) > 0\} \to 1 - e^{-\beta_\infty}$ as $t \to \infty$, where $\beta_\infty = \frac{\nu_1 Y_\infty}{\mu_1}$ is given in (3.5.1), and Y_∞ is the limiting value of the function $Y(t)$ as $t \to \infty$. Now $Y(t) = N_2 y(t)$, and, by Theorem 6.2.4.1, $y(t) \to 1 - \frac{1}{T_1 T_2}$ as $t \to \infty$. Using expression (6.1.3) for T_1, we find that $\beta_\infty = T_1 - \frac{1}{T_2}$. The steady-state prevalence is thus a function of T_1 and T_2. Denoting the function by Q_h (the subscript h refers to hermaphroditic helminthiasis), we have

$$Q_h(T_1,T_2) = 1 - e^{-(T_1 - \frac{1}{T_2})}. \qquad (7.2.1)$$

This relation shows that the steady-state prevalence is decreased by a decrease in either T_1 or T_2. For fixed $T_2 > 0$, the steady-state prevalence approaches the value 1 as $T_1 \to \infty$. On the other hand, for fixed $T_1 > 0$, the steady-state prevalence approaches the smaller value $1 - e^{-T_1}$ as $T_2 \to \infty$. It is also seen from (7.2.1) that the steady-state prevalence is decreased if T_1 is decreased and T_2 is increased so that the product $T_1 T_2$ remains constant at a value larger than 1. In contrast, the steady-state prevalence is increased if T_2 is decreased and T_1 is increased so that the product $T_1 T_2$ remains constant at a value larger than 1.

We use the word control to denote efforts at reduction in prevalence without necessarily aiming at eradication. In order to compare the efficiencies of controls based on T_1 and T_2, respectively, we compute

$$T_1 \frac{\partial Q_h(T_1,T_2)}{\partial T_1} = T_1[1 - Q_h(T_1,T_2)] \tag{7.2.2}$$

and

$$T_2 \frac{\partial Q_h(T_1,T_2)}{\partial T_2} = \frac{1}{T_2}[1 - Q_h(T_1,T_2)]. \tag{7.2.3}$$

Now $T_1T_2 > 1$. Hence the right-hand side of (7.2.2) is larger than the right-hand side of (7.2.3). We now compare the decrease in Q_h accompanying a small decrease in T_1 with the decrease in Q_h accompanying a proportionately equal decrease in T_2. Thus, let $\Delta T_1 < 0$ and take $\Delta T_2 = \frac{T_2}{T_1} \Delta T_1$. Put

$$\Delta_1 = Q_h(T_1+\Delta T_1,T_2) - Q_h(T_1,T_2) \text{ and}$$

$$\Delta_2 = Q_h(T_1,T_2 + \frac{T_2}{T_1} \Delta T_1) - Q_h(T_1,T_2).$$

Then

$$\Delta_1-\Delta_2 = \frac{\Delta T_1}{T_1}\left[T_1 \frac{\partial Q_h}{\partial T_1}(T_1+\theta_1\Delta T_1,T_2) - T_2 \frac{\partial Q_h}{\partial T_2}(T_1,T_2+\theta_2 \frac{T_2}{T_1} \Delta T_1)\right], \tag{7.2.4}$$

where $0 < \theta_1 < 1$ and $0 < \theta_2 < 1$. By the comment above we have

$$T_1 \frac{\partial Q_h}{\partial T_1}(T_1,T_2) - T_2 \frac{\partial Q_h}{\partial T_2}(T_1,T_2) > 0.$$

From the continuity of $\frac{\partial Q_h}{\partial T_1}$ and $\frac{\partial Q_h}{\partial T_2}$ we can take ΔT_1 so small that the expression within brackets in (7.2.4) is positive. It follows that $-\Delta_1 > -\Delta_2$ or $|\Delta_1| > |\Delta_2|$.

We interpret this result as saying that control of steady-state prevalence by decreasing T_1 is more efficient than control by decreasing T_2. We note also that the ratio of efficiencies is considerable if $T_1T_2 >> 1$, i.e. if the product of T_1 and T_2 is considerably larger than its threshold value.

7.3 The Proportion of Intermediate Hosts That Are Infected

In this section we study the proportion of intermediate hosts that are infected in the case $T_1T_2 > 1$ and $u_o \neq 0$. This proportion is given by the function $y(t)$, $t \geq 0$. We assume that a sufficiently long time has elapsed so that the proportion is close to its steady-state value. From Theorem 6.2.4.1, $y(t) \to 1 - \frac{1}{T_1T_2}$ as $t \to \infty$. The steady-state proportion is thus a function of T_1 and T_2. Denoting this function by I_h, we have

$$I_h(T_1,T_2) = 1 - \frac{1}{T_1T_2}. \qquad (7.3.1)$$

This relation shows that the function I_h is symmetric in its dependence on T_1 and T_2:

$$I_h(T_1,T_2) = I_h(T_2,T_1).$$

Holding either of the transmission factors fixed, the steady-state proportion approaches the value 1 as the other transmission factor $\to \infty$. It is also seen that the steady-state proportion I_h is decreased by a decrease of either T_1 or T_2. Also, I_h is constant on the hyperbolas T_1T_2 = constant in the T_1 - T_2 plane. Because of the above-mentioned symmetry, we conclude that control of the steady-state proportion of infected intermediate hosts by decreasing T_1 is as efficient as control by decreasing T_2.

7.4 The Public Health Factor

It was shown in the two preceding sections that a control program that works at reducing T_1 or T_2 (or both) leads to a reduction in prevalence and a reduction in the proportion of infected intermediate hosts. Another aspect of a control program is that it may improve public health conditions in the sense that the probability of any definitive host being infected during a short time interval is decreased. In this section we introduce and study a measure of public health conditions. We assume that $T_1T_2 > 1$ and $u_o \neq 0$.

From Chapter 3 we have that the probability of a given definitive host being infected during the time-interval $(t,t+\Delta t)$ is $\nu_1 Y(t)\,\Delta t + o(\Delta t)$. Now we have $Y(t) = N_2 y(t)$ and $T_1 = \frac{\nu_1 N_2}{\mu_1}$. Furthermore it follows from Theorem 6.2.4.1 that $y(t) \to 1 - \frac{1}{T_1 T_2}$ as $t \to \infty$. We therefore find that $\nu_1 Y(t) \to \nu_1 N_2 - \frac{\mu_1}{T_2}$ as $t \to \infty$. This expression is a measure

of steady-state public health conditions. We refer to it as a public health factor H_h. Noting that it is a function of $\nu_1 N_2$ and $\frac{T_2}{\mu_1}$ we thus define

$$H_h\left(\nu_1 N_2, \frac{T_2}{\mu_1}\right) = \nu_1 N_2 - \frac{\mu_1}{T_2}. \tag{7.4.1}$$

The public health factor is seen to decrease as a result of a decrease in either $\nu_1 N_2$ or $\frac{T_2}{\mu_1}$. For fixed $\nu_1 N_2$, the public health factor approaches $\nu_1 N_2$ as $\frac{T_2}{\mu_1} \to \infty$. For fixed $\frac{T_2}{\mu_1}$, the public health factor increases linearly with an increase in $\nu_1 N_2$.

In order to investigate control of the public health factor, we compute

$$\nu_1 N_2 \frac{\partial H_h\left(\nu_1 N_2, \frac{T_2}{\mu_1}\right)}{\partial(\nu_1 N_2)} = \nu_1 N_2 \tag{7.4.2}$$

and

$$\frac{T_2}{\mu_1} \frac{\partial H_h\left(\nu_1 N_2, \frac{T_2}{\mu_1}\right)}{\partial\left(\frac{T_2}{\mu_1}\right)} = \frac{\mu_1}{T_2}. \tag{7.4.3}$$

Now $T_1 = \frac{\nu_1 N_2}{\mu_1}$ and $T_1 T_2 > 1$. Hence the right-hand side of (7.4.2) is larger than the right-hand side of (7.4.3). We therefore have

$$\nu_1 N_2 \frac{\partial H_h\left(\nu_1 N_2, \frac{T_2}{\mu_1}\right)}{\partial(\nu_1 N_2)} > \frac{T_2}{\mu_1} \frac{\partial H_h\left(\nu_1 N_2, \frac{T_2}{\mu_1}\right)}{\partial\left(\frac{T_2}{\mu_1}\right)}.$$

Using an argument similar to that in Section 7.2, we arrive at the following interpretation of this result: Control of the public health factor through a reduction of $\nu_1 N_2$ is more efficient than control through reduction of $\frac{T_2}{\mu_1}$. The ratio of efficiencies is considerable if $T_1 T_2 >> 1$.

Chapter 8

Model for Schistosomiasis

8.1 Model Formulation

In this section we formulate a model for the transmission of schistosomiasis. We use the following notation:

N_1 = the total number of definitive hosts (assumed constant);

$\xi_{F_k}(t)$ = the number of mature female parasites alive at time t inside definitive host number k, $k=1,2,\ldots,N_1$, $t \geq 0$;

$f_k = \xi_{F_k}(0)$ = the initial number of mature female parasites alive inside definitive host number k, $k=1,2,\ldots,N_1$;

$P_{k;f}(t) = P\{\xi_{F_k}(t) = f\}$, $k=1,2,\ldots,N_1$, $f=0,1,2,\ldots$, $t \geq 0$;

$\xi_{M_k}(t)$ = the number of mature male parasites alive at time t inside definitive host number k, $k=1,2,\ldots,N_1$, $t \geq 0$;

$m_k = \xi_{M_k}(0)$ = the initial number of mature male parasites alive inside definitive host number k, $k=1,2,\ldots,N_1$;

$Q_{k;m}(t) = P\{\xi_{M_k}(t) = m\}$, $k=1,2,\ldots,N_1$, $m=0,1,2,\ldots$, $t \geq 0$;

$\xi_{P_k}(t)$ = the number of paired mature parasites alive at time t inside definitive host number k, $k=1,2,\ldots,N_1$, $t \geq 0$;

$S_{k;p}(t) = P\{\xi_{P_k}(t) = 2p\}$, $k=1,2,\ldots,N_1$, $p=0,1,2,\ldots$, $t \geq 0$;

N_2 = the total number of intermediate hosts (assumed constant);

$\eta(t)$ = the number of infected intermediate hosts at time $t \geq 0$;

$m_2 = N_2 y_o = \eta(0)$ = the initial number of infected intermediate hosts;

$\Pi_i(t) = P\{\eta(t) = i\}$, $i=0,1,2,\ldots,N_2$, $t \geq 0$.

We construct a model for the distributions of the $2N_1 + 1$ random variables

$$\xi_{F_1}(t),\ \xi_{F_2}(t),\ldots,\ \xi_{F_{N_1}}(t),\ \xi_{M_1}(t),\ \xi_{M_2}(t),\ldots,\ \xi_{M_{N_1}}(t),$$

and $\eta(t)$. For each $k=1,2,\ldots,N_1$ we take $\xi_{F_k}(t)$ and $\xi_{M_k}(t)$ to be independently distributed random variables. Each is the number of individuals in a non-homogeneous immigration-death process with immigration rate $\frac{1}{2}\nu_1 E\eta(t)$ and death rate μ_1 per individual. Furthermore, $\eta(t)$ is taken to be the number of individuals among N_2 in a non-homogeneous infection-recovery process with infection rate $\frac{1}{2}\nu_2 E\sum_{k=1}^{N_1}\xi_{P_k}(t)$ per uninfected individual and recovery rate μ_2 per individual. The initial values of each of the $2N_1 + 1$ random variables are assumed to be given.

Note now that

$$\{\xi_{P_k}(t) = 2p\}$$

$$= \bigcup_{j=0}^{\infty} \left[\{\xi_{F_k}(t)=p\} \cap \{\xi_{M_k}(t)=p+j\}\right] \cup \bigcup_{i=1}^{\infty} \left[\{\xi_{M_k}(t)=p\} \cap \{\xi_{F_k}(t)=p+i\}\right].$$

Now all events inside brackets in the above expression are mutually exclusive. Furthermore, the random variables $\xi_{F_k}(t)$ and $\xi_{M_k}(t)$ are by hypothesis independent. Hence we get

$$S_{k;p}(t) = P_{k;p}(t) \sum_{j=0}^{\infty} Q_{k;p+j}(t) + Q_{k;p}(t) \sum_{i=1}^{\infty} P_{k;p+i}(t).$$

Using arguments in Chapters 3 and 4 we arrive at the following system of differential-difference equations with initial conditions for the unknown functions $P_{k;f}(t)$, $Q_{k;m}(t)$, and $\Pi_i(t)$:

$$P'_{k;f}(t) = \frac{1}{2} \nu_1 \sum_{i=1}^{N_2} i\Pi_i(t) P_{k;f-1}(t)$$

$$- \left[\frac{1}{2} \nu_1 \sum_{i=1}^{N_2} i\Pi_i(t) + f\mu_1\right] P_{k;f}(t)$$

$$+ (f+1) \mu_1 P_{k;f+1}(t),$$

$$k=1,2,\ldots,N_1, \quad f=0,1,2,\ldots, \quad t \geq 0, \qquad (8.1.1)$$

$$P_{k;f}(0) = \begin{cases} 1, & f = f_k, \\ 0, & f \neq f_k; \end{cases}$$

$$Q_{k;m}'(t) = \frac{1}{2}\nu_1 \sum_{i=1}^{N_2} i\Pi_i(t)\, Q_{k;m-1}(t)$$

$$- \left[\frac{1}{2}\nu_1 \sum_{i=1}^{N_2} i\Pi_i(t) + m\mu_1\right] Q_{k;m}(t)$$

$$+ (m+1)\mu_1 Q_{k;m+1}(t),$$

$$k=1,2,\ldots,N_1,\ m=0,1,2,\ldots,\ t \geq 0, \tag{8.1.2}$$

$$Q_{k;m}(0) = \begin{cases} 1, & m = m_k, \\ 0, & m \neq m_k; \end{cases}$$

$$\Pi_i'(t) = \nu_2 \sum_{k=1}^{N_1} \sum_{p=1}^{\infty} p \left[P_{k;p}(t) \sum_{j=0}^{\infty} Q_{k;p+j}(t) + Q_{k;p}(t) \sum_{i=1}^{\infty} P_{k;p+i}(t)\right](N_2-i+1)\Pi_{i-1}(t)$$

$$- \left\{\nu_2 \sum_{k=1}^{N_1} \sum_{p=1}^{\infty} p\left[P_{k;p}(t) \sum_{j=0}^{\infty} Q_{k;p+j}(t) + Q_{k;p}(t) \sum_{i=1}^{\infty} P_{k;p+i}(t)\right](N_2-i) + i\mu_2\right\} \Pi_i(t)$$

$$+ (i+1)\,\mu_2\, \Pi_{i+1}(t),$$

$$i = 0,1,2,\ldots,N_2,\ t \geq 0, \tag{8.1.3}$$

$$\Pi_i(0) = \begin{cases} 1, & i = m_2 = N_2 y_o, \\ 0, & i \neq m_2. \end{cases}$$

Here, $P_{k;-1}(t) = Q_{k;-1}(t) = 0$, $k=1,2,\ldots,N_1$, $t \geq 0$,

and $\Pi_{-1}(t) = \Pi_{N_2+1}(t) = 0$, $t \geq 0$.

Any non-negative functions $P_{k;f}(t)$, $Q_{k;m}(t)$, and $\Pi_i(t)$ that satisfy the above equations for all $t \geq 0$ and the above initial conditions and

$$\sum_{f=0}^{\infty} P_{k;f}(t) \leq 1, \qquad \sum_{m=0}^{\infty} Q_{k;m}(t) \leq 1,$$

$$\sum_{p=1}^{\infty} p\left[P_{k;p}(t) \sum_{j=0}^{\infty} Q_{k;p+j}(t) + Q_{k;p}(t) \sum_{i=1}^{\infty} P_{k;p+i}(t)\right] < \infty ,$$

$$k=1,2,\ldots, N_1, \; t \geq 0, \; \sum_{i=0}^{N_2} \Pi_i(t) = 1, \; t \geq 0,$$

are referred to as a solution of the model formulated above.

The parameters ν_1 and ν_2 can now be given the following interpretations: ν_1 is the rate of emission of cercariae by each infected intermediate host multiplied by the probability that a cercaria is successful in penetrating a given permanent host. Similarly, ν_2 is the rate of egg production of each mature paired female parasite multiplied by the probability that an egg that has been

laid leads to infection of a given non-infected intermediate host. We note also that $\nu_1 N_1$ is the rate of emission of successful cercariae by each infected intermediate host, and that $\nu_2 N_2$ would be the rate of emission of successful miracidia by each mature paired female parasite if we should recognize a miracidium as successful whenever it penetrates either a non-infected or an infected intermediate host.

The definitions of the transmission factors T_1 and T_2 that were given in (6.1.3) and (6.1.4), respectively, are the same for both hermaphroditic helminthiasis and schistosomiasis.

The model formulated in this section has both stochastic and deterministic features. The deterministic aspect derives from the assumptions that immigration rates and infection rates are proportional to the expectations of $\eta(t)$ and $\frac{1}{2} \sum_{k=1}^{N_1} \xi_{P_k}(t)$, respectively.

In the next section (8.2) we formulate and study an initial-value problem, derived from the model in this section. The results in Section 8.2 are used in Section 8.3 to establish existence and uniqueness of the solution of the model. They are also used in the study of the epidemiology of schistosomiasis in Chapter 9.

8.2 An Initial-Value Problem

For the discussion in this section we assume that a solution exists of the model formulated in Section 8.1.

We define the four functions $Y(t)$, $y(t)$, $Z(t)$, $z(t)$ by the following relations:

$$Y(t) = N_2 y(t) = \sum_{i=1}^{N_2} i\Pi_i(t) \tag{8.2.1}$$

and

$$Z(t) = N_1 z(t)$$

$$= \sum_{k=1}^{N_1} \sum_{p=1}^{\infty} p\left[P_{k;p}(t) \sum_{j=0}^{\infty} Q_{k;p+j}(t) + Q_{k;p}(t) \sum_{i=1}^{\infty} P_{k;p+i}(t)\right]. \tag{8.2.2}$$

Thus, $Y(t)$ is the expected value of the random variable $\eta(t)$, while $Z(t)$ is the sum over all N_1 definitive hosts of the expected number of paired female parasites.

For any $k=1,2,\ldots,N_1$, consider now the system of differential-difference equations with initial data given in (8.1.1). Since the functions $\Pi_i(t)$ appearing in (8.2.1) are solutions of the model in Section 8.1, they are all non-negative, bounded, and continuous. The function $Y(t)$ in (8.2.1) has therefore the same three properties. The systems (8.1.1) and (3.1.7) are now seen to be formally identical. In Chapter 3 we showed that system (3.1.7) has a unique solution. The same conclusion therefore holds with regard to system (8.1.1). A similar comparison between systems (8.1.2) and (3.1.8) leads to the conclusion

that system (8.1.2) has a unique solution. In Chapter 3 we proceeded to derive an expression for the expectation of the random variable $\xi_P(t) = 2(\xi_F(t) \wedge \xi_M(t))$. From Theorem 3.4.1 we therefore conclude

$$\sum_{p=1}^{\infty} p\left[P_{k;p}(t) \sum_{j=0}^{\infty} Q_{k;p+j}(t) + Q_{k;p}(t) \sum_{i=1}^{\infty} P_{k;p+i}(t)\right]$$

$$= \frac{1}{2}\psi(\beta(t)) + \frac{1}{2} G(\alpha(t), \beta(t); f_k, m_k), \tag{8.2.3}$$

where the functions α, β, ψ, and G are defined in (3.2.1), (3.2.2), (3.4.2), and (3.3.16), respectively. By inserting expression (8.2.3) into (8.2.2) we find

$$z(t) = \frac{1}{2}\psi(\beta(t)) + \frac{1}{2} F(\alpha(t), \beta(t)), \tag{8.2.4}$$

where

$$F(\alpha,\beta) = \frac{1}{N_1} \sum_{k=1}^{N_1} G(\alpha, \beta; f_k, m_k). \tag{8.2.5}$$

We note that the function F depends on all the initial values f_k, m_k for $k=1,2,\ldots,N_1$.

Now consider the system of differential-difference equations with initial data given in (8.1.3). From relations (8.2.4) and (8.2.5) and Lemmas 3.4.1(i) and 3.6.1(iii) it follows that the functions $z(t)$ and $Z(t) = N_2 z(t)$ are non-negative, bounded, and continuous. The systems (8.1.3) and (4.1.1) are now seen to be formally identical,

since the function Z(t) of (8.2.2) in the former system has the same properties as the function X(t) in the latter system. In Chapter 4 we showed that system (4.1.1) has a unique solution, and we found, in (4.2.5) and (4.2.6), an initial-value problem that is satisfied by the expectation of the random variable $\eta(t)$ whose distribution is given by the functions $\Pi_i(t)$. Hence the function Y(t) of (8.2.1) satisfies the same initial-value problem with the function X(t) replaced by Z(t). Recalling that $Y(t) = N_2 y(t)$, $Z(t) = N_1 z(t)$, and $T_2 = \frac{\nu_2 N_1}{\mu_2}$, and using (8.2.4), we get

$$y'(t) = \frac{1}{2} T_2 \mu_2 [\psi(\beta(t)) + F(\alpha(t), \beta(t))] (1-y(t)) - \mu_2 y(t). \quad (8.2.6)$$

At t = 0 we have from (8.2.1)

$$y(0) = y_o. \quad (8.2.7)$$

From (3.2.1) and (3.2.2) it follows that

$$\alpha'(t) = -\mu_1 \alpha(t), \quad (8.2.8)$$

$$\beta'(t) = -\mu_1 \beta(t) + T_1 \mu_1 y(t), \quad (8.2.9)$$

$$\alpha(0) = 1, \quad (8.2.10)$$

$$\beta(0) = 0. \quad (8.2.11)$$

In the sequel, the following notation is used:

$$w(t) = \begin{pmatrix} \alpha(t) \\ \beta(t) \\ y(t) \end{pmatrix},$$

$$h(w) = \begin{pmatrix} h_1(w) \\ h_2(w) \\ h_3(w) \end{pmatrix} = \begin{pmatrix} -\mu_1\alpha \\ -\mu_1\beta + T_1\mu_1 y \\ \frac{1}{2} T_2\mu_2[\psi(\beta) + F(\alpha,\beta)] (1-y) - \mu_2 y \end{pmatrix},$$

$$w_o = \begin{pmatrix} 1 \\ 0 \\ y_o \end{pmatrix}.$$

The initial-value problem (8.2.6)-(8.2.11) can then also be written in the abbreviated form

$$\begin{cases} w' = h(w), \\ w(0) = w_o. \end{cases}$$

This initial-value problem is called Problem (8.2). We devote the following four subsections (8.2.1-8.2.4) to an investigation of properties of the solution of Problem (8.2).

8.2.1 Existence, Uniqueness, and Boundedness of the Solution

In this subsection we prove a theorem concerning global existence, uniqueness, and boundedness of the solution of Problem (8.2). The theorem is preceded by two lemmas. For the first of these we introduce the following notation:

$$\xi(t) = \begin{pmatrix} \beta(t) \\ \\ y(t) \end{pmatrix},$$

$$\chi(\xi) = \begin{pmatrix} \chi_1(\xi) \\ \\ \chi_2(\xi) \end{pmatrix} = \begin{pmatrix} -\mu_1\beta + T_1\mu_1 y \\ \\ \frac{1}{2}\, T_2\mu_2\, \psi(\beta)(1-y) - \mu_2 y \end{pmatrix},$$

$$C_o = \{(\beta,y): 0 \leq \beta \leq T_1,\ 0 \leq y \leq 1\}. \qquad (8.2.1.1)$$

Lemma 8.2.1.1

The set C_o is invariant with respect to the system $\xi' = \chi(\xi)$.

Proof:

By Proposition 5.2 it suffices to show that every point on ∂C_o is an ingress point of C_o. We consider several subsets of the boundary ∂C_o. In each case we use $\overline{\xi} = \begin{pmatrix} \overline{\beta} \\ \overline{y} \end{pmatrix}$ to denote a point belonging to the subset of ∂C_o under consideration. We then consider the problem

$$\begin{cases} \xi' = \chi(\xi), \\ \\ \xi(0) = \overline{\xi}. \end{cases}$$

Since the Jacobian matrix $J = \frac{\partial \chi}{\partial \xi}$ is continuous, this problem has a unique solution in some time-interval $-\varepsilon < t < \varepsilon$, $\varepsilon > 0$. We denote the solution by $\xi(t) = \begin{pmatrix} \beta(t) \\ y(t) \end{pmatrix}$. A tangent to the solution is then given by the vector $T(t) = (\beta'(t), y'(t))$, and $T(0) = (\chi_1(\overline{\xi}), \chi_2(\overline{\xi}))$. We use N to denote a normal to ∂C_o directed into C_o. Consider now the following four lines:

1) $\bar{\beta} = 0$, $\bar{y} > 0$.

Here, $N = (1,0)$, and

$T(0) = (T_1\mu_1\bar{y},\ \chi_2(\bar{\xi}))$.

Hence $N\cdot T(0) = T_1\mu_1\bar{y} > 0$.

2) $\bar{\beta} = T_1$, $\bar{y} < 1$.

Here, $N = (-1,0)$, and

$T(0) = (-\mu_1 T_1 + T_1\mu_1\bar{y},\ \chi_2(\bar{\xi}))$.

Hence $N\cdot T(0) = \mu_1 T_1 - T_1\mu_1\bar{y} > 0$.

3) $\bar{\beta} > 0$, $\bar{y} = 0$.

Here, $N = (0,1)$, and

$T(0) = (\chi_1(\bar{\xi}),\ \frac{1}{2}\, T_2\mu_2\psi(\bar{\beta}))$.

Hence $N\cdot T(0) = \frac{1}{2}\, T_2\mu_2\psi(\bar{\beta}) > 0$,

using Lemma 3.4.1(i).

4) $\bar{y} = 1$.

Here, $N = (0,-1)$, and

$T(0) = (\chi_1(\bar{\xi}),\ -\mu_2)$.

Hence $N\cdot T(0) = \mu_2 > 0$.

From these results and Proposition 5.3 we conclude that all points on ∂C_o, except possibly $(0,0)$, $(0,1)$, $(T_1,0)$, and $(T_1,1)$, are ingress points of $\overset{\circ}{C}_o$. By also applying Proposition 5.1 (ii) we find $(0,1)$ and $(T_1,0)$ to be ingress points of $\overset{\circ}{C}_o$.

For the point $\bar{\beta} = T_1$, $\bar{y} = 1$, we have

$$\beta'(0) = 0,$$
$$y'(0) = -\mu_2 < 0,$$
$$\beta''(0) = -T_1\mu_1\mu_2 < 0.$$

Hence the function $\beta(t)$ has a local maximum at $t = 0$. We conclude that $(T_1,1)$ is an ingress point of the set $\overset{\circ}{C}_o$.

For the point $\bar{\beta} = \bar{y} = 0$, the solution is $\xi(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $t \geq 0$. Hence $\xi(t) \in \partial C_o$, $t \geq 0$, which shows that $(0,0)$ is an ingress point of the closed set C_o. QED

For the next lemma, we introduce the notation

$$C = \{(\alpha,\beta,y): \ 0 \leq \alpha \leq 1, \ 0 \leq \beta \leq T_1, \ 0 \leq y \leq 1\}. \qquad (8.2.1.2)$$

Lemma 8.2.1.2 The set C is invariant with respect to the system $w' = h(w)$ of Section 8.2.

Proof: By Proposition 5.2 it suffices to show that every point on ∂C is an ingress point of C. We consider several subsets of the boundary ∂C. In each case we use $\bar{w} = \begin{bmatrix} \bar{\alpha} \\ \bar{\beta} \\ \bar{y} \end{bmatrix}$ to denote a point belonging to the subset of ∂C under study. We then consider the problem

$$\begin{cases} w' = h(w), \\ w(0) = \bar{w}. \end{cases}$$

Since the Jacobian matrix $J = \dfrac{\partial h}{\partial w}$ is continuous, this problem has a unique solution in some time-interval $-\varepsilon < t < \varepsilon$, $\varepsilon > 0$. We denote the solution by $w(t) = \begin{bmatrix} \alpha(t) \\ \beta(t) \\ y(t) \end{bmatrix}$. A tangent to the solution is then given by the vector $T(t) = (\alpha'(t), \beta'(t), y'(t))$, and $T(0) = (h_1(\bar{w}), h_2(\bar{w}), h_3(\bar{w}))$. We use N to denote a normal to ∂C directed into C.

We now consider the following subsets of ∂C:

1) $\bar{\alpha} = 0$, $0 \leq \bar{\beta} \leq T_1$, $0 \leq \bar{y} \leq 1$. Here, our problem has a solution with $\alpha(t) = 0$ for $t \geq 0$ while $\beta(t)$ and $y(t)$ satisfy the related initial-value problem

$$\begin{cases} \xi' = \chi(\xi), \\ \xi(0) = \bar{\xi}, \end{cases}$$

of Lemma 8.2.1.1. From Lemma 8.2.1.1 we conclude that our problem has a solution $w(t)$ such that $w(t) \in C_o \subset \partial C$ for $t \geq 0$. Therefore, each point $\bar{w}$ in the subset of ∂C considered here is an ingress point of the closed set C.

2) $\bar{\alpha} = 1$.

Here, $N = (-1, 0, 0)$, and

$T(0) = (-\mu_1, h_2(\bar{w}), h_3(\bar{w}))$.

Hence $N \cdot T(0) = \mu_1 > 0$.

3) $\bar{\beta} = 0$, $\bar{y} > 0$.

Here, $N = (0, 1, 0)$, and

$T(0) = (h_1(\bar{w}), T_1\mu_1\bar{y}, h_3(\bar{w}))$.

Hence $N \cdot T(0) = T_1\mu_1\bar{y} > 0$.

4) $\bar{\beta} = T_1$, $\bar{y} < 1$.

Here, $N = (0, -1, 0)$, and

$T(0) = (h_1(\bar{w}), -\mu_1 T_1(1-\bar{y}), h_3(\bar{w}))$.

Hence $N \cdot T(0) = \mu_1 T_1 (1-\bar{y}) > 0$.

5) $\bar{\beta} > 0$, $\bar{y} = 0$.

Here, $N = (0, 0, 1)$, and

$T(0) = (h_1(\bar{w}), h_2(\bar{w}), \frac{T_2\mu_2}{2}[\psi(\bar{\beta}) + F(\bar{\alpha},\bar{\beta})])$.

Hence $N \cdot T(0) = \frac{T_2\mu_2}{2}[\psi(\bar{\beta}) + F(\bar{\alpha},\bar{\beta})] > 0$,

using Lemmas 3.4.1(i) and 3.6.1(iii) and relation (8.2.5).

6) $\bar{y} = 1$.

Here, $N = (0, 0, -1)$, and

$T(0) = (h_1(\bar{w}), h_2(\bar{w}), -\mu_2)$.

Hence $N \cdot T(0) = \mu_2 > 0$.

From these results and Proposition 5.3 we conclude that all points interior to the five sides of C for which $\alpha > 0$ are ingress points of $\overset{\circ}{C}$. By also applying Proposition 5.1 (ii) we find that all points on edges and corners of C for which $\alpha > 0$, except possibly points on

$$L_1 = \{(\alpha,\beta,y): \ 0 < \alpha \leq 1, \ \beta = T_1, \ y = 1\}$$

and

$$L_2 = \{(\alpha,\beta,y): \ 0 < \alpha \leq 1, \ \beta = y = 0\},$$

are ingress points of the set $\overset{\circ}{C}$.

For any point $\bar{w} \in L_1$ we have

$$\alpha'(0) = -\mu_1\bar{\alpha} < 0,$$

$$\beta'(0) = 0,$$

$$y'(0) = -\mu_2 < 0,$$

$$\beta''(0) = -T_1\mu_1\mu_2 < 0.$$

Hence the function $\beta(t)$ has a local maximum at $t = 0$. We conclude that each point $\bar{w} \in L_1$ is an ingress point of the set $\overset{\circ}{C}$.

Finally, we consider $\bar{w} \in L_2$. Here we distinguish two subcases:

A: $f_k \wedge m_k > 0$ for at least one $k = 1,2,\ldots,N_1$.

B: $f_k \wedge m_k = 0$ for all $k = 1,2,\ldots,N_1$.

In subcase A we have

$$\alpha'(0) = -\mu_1\bar{\alpha} < 0,$$

$$\beta'(0) = 0,$$

$$y'(0) = \tfrac{1}{2} T_2\mu_2 F(\bar{\alpha},0) > 0,$$

since $\psi(0) = 0$ from (3.4.2) and $F(\bar{\alpha},0) > 0$ from Lemma 3.6.1 (v) and (8.2.5). Also,

$$\beta''(0) = \tfrac{1}{2} T_1T_2\mu_1\mu_2 F(\bar{\alpha},0) > 0.$$

Hence the function $\beta(t)$ has a local minimum at $t = 0$. We conclude that $\bar{w}$ is an ingress point of the open set $\overset{\circ}{C}$.

In subcase B we have

$$\alpha'(0) = -\mu_1\bar{\alpha} < 0,$$

$$\beta'(0) = 0,$$

$$y'(0) = 0,$$

since $\psi(0) = 0$ from (3.4.2) and $F(\bar{\alpha},0) = 0$ from Lemma 3.6.1(iv) and (8.2.5). The solution is found explicitly to be

$$w(t) = \begin{Bmatrix} \bar{\alpha}\, e^{-\mu_1 t} \\ 0 \\ 0 \end{Bmatrix}, \quad t \geq 0.$$

Hence $w(t) \in \partial C$ for $t \geq 0$. The point $\bar{w}$ is therefore an ingress point of the closed set C. QED

We now apply the preceding lemma in the proof of the following theorem:

Theorem 8.2.1.1 Problem (8.2) has a unique solution $w(t) = \begin{Bmatrix} \alpha(t) \\ \beta(t) \\ y(t) \end{Bmatrix}$. The solution exists for all $t \geq 0$. The solution components have the following bounds: $0 \leq \alpha(t) \leq 1$, $0 \leq \beta(t) \leq T_1$, $0 \leq y(t) \leq 1$, $t \geq 0$.

Proof:

Let the rectangular parallelepiped C be defined as in (8.2.1.2). From Lemma 8.2.1.2 we conclude that the set C is invariant with respect to the system $w' = h(w)$. The conclusion of the theorem now follows from Definition 5.1 since the initial point w_o belongs to the set C. QED

8.2.2 The Critical Points

In this subsection we study the critical points of Problem (8.2). A point is said to be a critical point of this problem if it satisfies two criteria:

1) It is a critical point of the system $w' = h(w)$ of Section 8.2.

2) It belongs to the invariant set C defined in (8.2.1.2).

The main results are contained in a theorem which is preceded by eight lemmas. We use the notation $R^+ = \{x : x > 0\}$ to denote the set of positive real numbers.

Lemma 8.2.2.1

Let $f : R^+ \to R^+$ be defined by $f(x) = \frac{e^x - I_o(x)}{I_1(x)}$. Then f has the following properties:

1) $f(x) > 2$,

2) $f'(x) > 0$,

3) $f(x) \sim 2$, $(x \to 0)$,

4) $f(x) \sim \sqrt{2\pi x}$, $(x \to \infty)$.

Proof: The modified Bessel functions $I_n(x)$ are used here. We refer to their properties as quoted in (3.3.1) - (3.3.7).

From (3.3.1) follows $I_n(x) > 0$ for $x > 0$ and $n = 0,1,2,\ldots$ Hence we get from (3.3.5)

$$e^x > I_o(x) + 2\, I_1(x), \quad x > 0,$$

and

$$f(x) = \frac{e^x - I_o(x)}{I_1(x)} > 2, \quad x > 0,$$

which proves the first statement. Next,

$$f'(x) = \frac{g(x)}{I_1^2(x)},$$

where

$$g(x) = [e^x - I_1(x)]\, I_1(x) - [e^x - I_0(x)]\, \tfrac{1}{2}\, [I_0(x) + I_2(x)].$$

Here we have applied (3.3.4) and (3.3.2). The functions $f'(x)$ and $g(x)$ have the same sign. Now, from (3.3.1),

$$I_0(0) = 1,$$

$$I_n(0) = 0, \qquad n = 1,2,\ldots$$

Hence $g(0) = 0$. In order to prove that $f'(x) > 0$ for $x > 0$ it suffices to show that $g'(x) > 0$ for $x > 0$. By applying (3.3.4) and (3.3.2) we find the following expression for $g'(x)$ after algebraic simplification:

$$g'(x) = \tfrac{1}{4}\, [e^x - I_0(x)]\, [I_1(x) - I_3(x)]$$

$$+ \tfrac{1}{2}\, [I_0(x) - I_2(x)]\, I_1(x).$$

Here we apply (3.3.3) to get

$$g'(x) = \frac{1}{x}\, [e^x - I_0(x)]\, I_2(x) + \frac{1}{x}\, I_1^2(x).$$

From $I_n(x) > 0$ for $x > 0$ and (3.3.5) we find $g'(x) > 0$. This proves the second statement.

From (3.3.1) we find for small x:

$$e^x - I_o(x) = x + O(x^2)$$

and

$$I_1(x) = \frac{x}{2} + O(x^3).$$

Hence

$$f(x) = 2 + O(x),$$

which proves the third statement.

Finally, (3.3.6) gives

$$e^x - I_o(x) \sim e^x, \qquad (x \to \infty),$$

and

$$I_1(x) \sim \frac{e^x}{\sqrt{2\pi x}}, \qquad (x \to \infty).$$

Hence

$$f(x) \sim \sqrt{2\pi x}, \qquad (x \to \infty).$$

This completes the proof of the lemma. QED

The function f defined above is related to the function ψ of (3.4.2), and to its derivative ψ', as follows:

$$f(x) - 1 = \frac{\psi(x)}{xe^{-x}I_1(x)}, \qquad (8.2.2.1)$$

$$f(x) = \frac{\psi'(x)}{e^{-x}I_1(x)}. \qquad (8.2.2.2)$$

Lemma 8.2.2.2

Let $\overline{T}_1$: $R^+ \to R^+$ be defined by

$$\overline{T}_1(\beta) = \beta f(\beta),$$

where f is defined in Lemma 8.2.2.1. Then $\overline{T}_1'(\beta) > 2$ for $\beta > 0$, and the inverse $\beta_{ml} = \overline{T}_1^{-1}$ exists. It satisfies the inequality $0 < \beta_{ml}(T_1) < \frac{T_1}{2}$, $T_1 > 0$.

Proof:

$$\overline{T}_1'(\beta) = \beta f'(\beta) + f(\beta).$$

By Lemma 8.2.2.1,

$$\overline{T}_1'(\beta) > 2 \quad \text{for } \beta > 0.$$

Hence $\overline{T}_1$ is injective. To prove that $\overline{T}_1$ is surjective, note that by Lemma 8.2.2.1

$$\overline{T}_1(\beta) \sim 2\beta, \ (\beta \to 0), \qquad (8.2.2.3)$$

and

$$\overline{T}_1(\beta) \sim \beta\sqrt{2\pi\beta}, \ (\beta \to \infty). \qquad (8.2.2.4)$$

By continuity of the function $\bar{T}_1$, we conclude that it maps R^+ onto R^+. Hence $\bar{T}_1$ is bijective and its inverse β_{m1} exists.

From $\bar{T}_1'(\beta) > 2$ for $\beta > 0$ we get

$$0 < \beta_{m1}'(T_1) < \frac{1}{2}, \; T_1 > 0.$$

Using continuity we extend the function β_{m1} so that $\beta_{m1}(0) = 0$. From the mean value theorem we conclude $0 < \beta_{m1}(T_1) < \frac{1}{2} T_1$ for $T_1 > 0$. QED

The asymptotic expressions for the function $\bar{T}_1$ exhibited in the proof above show that the inverse function β_{m1} has the following asymptotic behavior:

$$\beta_{m1}(T_1) \sim \frac{1}{2} T_1, \qquad (T_1 \to 0), \tag{8.2.2.5}$$

$$\beta_{m1}(T_1) \sim \frac{T_1^{2/3}}{(2\pi)^{1/3}}, \quad (T_1 \to \infty). \tag{8.2.2.6}$$

A plot of the function β_{m1} appears in Figure 8.2.2.1. In this plot, the scales are logarithmic. Thus, the asymptotic expressions given above appear as straight line segments.

Lemma 8.2.2.3

Let $\bar{T}_2 : R^+ \to R^+$ be defined by

$$\bar{T}_2(\beta) = \frac{2}{f(\beta)-1} \frac{1}{\psi(\beta)}.$$

Then $\bar{T}_2'(\beta) < 0$ for $\beta > 0$, and its inverse $\beta_{m2} = \bar{T}_2^{-1}$ exists.

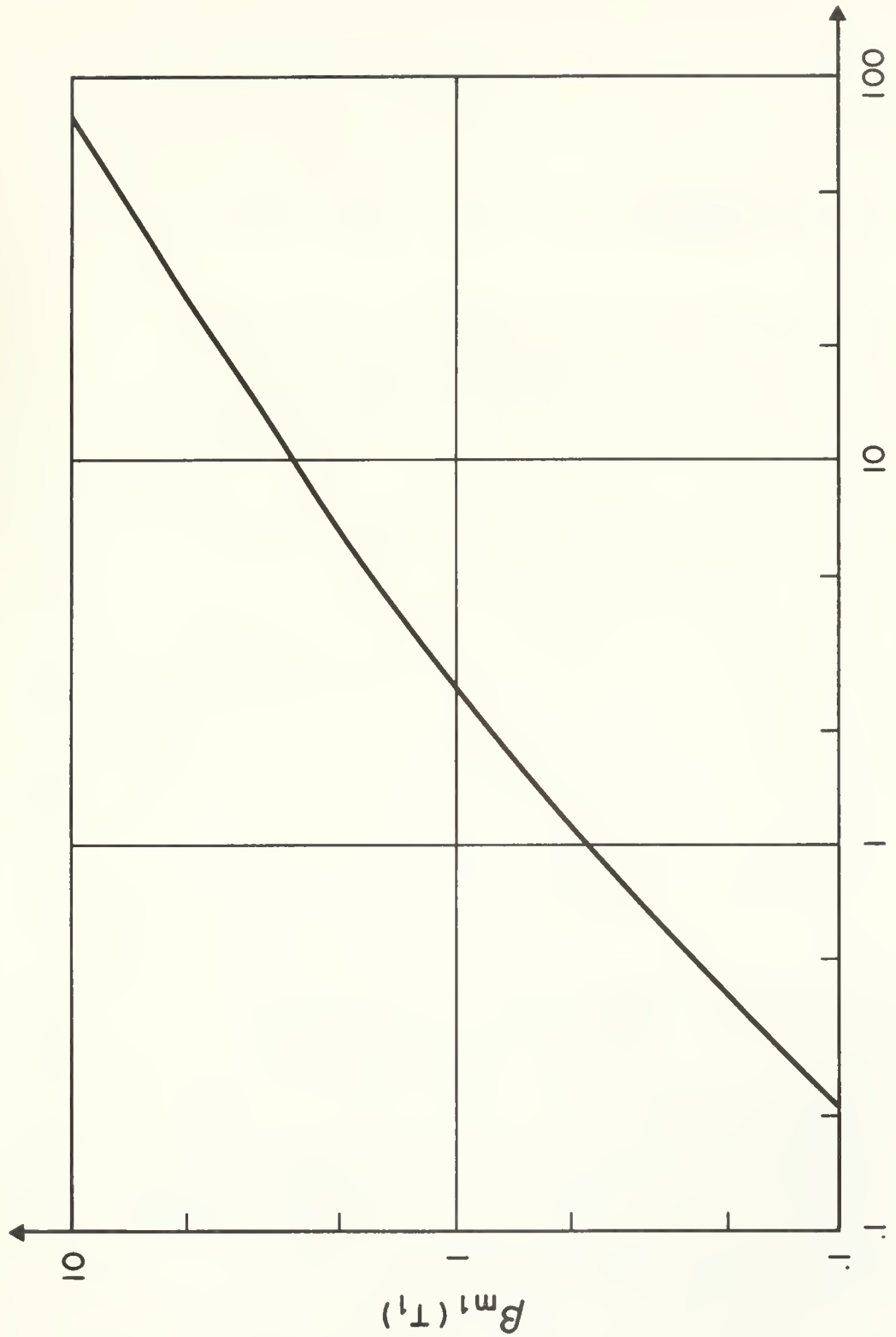

FIGURE 8.2.2.1

Proof:

$$\bar{T}_2'(\beta)$$

$$= - \frac{2}{[f(\beta)-1]^2[\psi(\beta)]^2} [f'(\beta)\psi(\beta) + (f(\beta)-1)\psi'(\beta)].$$

Now for $\beta > 0$ we have $f'(\beta) > 0$ and $f(\beta) > 2$ by Lemma 8.2.2.1, while $\psi(\beta) > 0$ and $\psi'(\beta) > 0$ follows from Lemma 3.4.1. Hence $\bar{T}_2'(\beta) < 0$ for $\beta > 0$. This proves that the function $\bar{T}_2$ is injective.

For small β we have $f(\beta) \sim 2$ by Lemma 8.2.2.1 and $\psi(\beta) \sim \frac{1}{2} \beta^2$ by Lemma 3.4.1. Hence

$$\bar{T}_2(\beta) \sim \frac{4}{\beta^2}, \qquad (\beta \to 0). \tag{8.2.2.7}$$

For large β, $f(\beta) \sim \sqrt{2\pi\beta}$ by Lemma 8.2.2.1 and $\psi(\beta) \sim \beta$ by Lemma 3.4.1. Hence

$$\bar{T}_2(\beta) \sim \sqrt{\frac{2}{\pi}} \frac{1}{\beta^{3/2}}, \quad (\beta \to \infty). \tag{8.2.2.8}$$

We conclude that $\lim_{\beta \downarrow 0} \bar{T}_2(\beta) = \infty$ and $\lim_{\beta \to \infty} \bar{T}_2(\beta) = 0$. By continuity of the function $\bar{T}_2$, it follows that it is surjective and hence bijective. QED

The asymptotic expressions for the function $\bar{T}_2$ exhibited in the proof above show that the inverse function β_{m2} has the following asymptotic behavior:

$$\beta_{m2}(T_2) \sim \sqrt[3]{\frac{2}{\pi}} \frac{1}{T_2^{2/3}}, \qquad (T_2 \to 0), \tag{8.2.2.9}$$

$$\beta_{m2}(T_2) \sim \frac{2}{\sqrt{T_2}}, \qquad (T_2 \to \infty). \tag{8.2.2.10}$$

A plot of the function β_{m2} appears in Figure 8.2.2.2. This function is used together with function $\bar{T}_1$ of Lemma 8.2.2.2 in the following lemma:

Lemma 8.2.2.4

Let $T_{10}: R^+ \to R^+$ be defined by

$$T_{10} = \bar{T}_1 \circ \beta_{m2}.$$

Then $T_{10}'(T_2) < 0$ for $T_2 > 0$.

Proof:

Let $T_2 > 0$. Then

$$T_{10}'(T_2) = \bar{T}_1'(\beta_{m2}(T_2)) \cdot \beta_{m2}'(T_2).$$

The first factor is positive by Lemma 8.2.2.2, and the second factor is negative by Lemma 8.2.2.3. QED

Asymptotic expressions for the function T_{10} are found from relations (8.2.2.3), (8.2.2.4), (8.2.2.9), and (8.2.2.10):

$$T_{10}(T_2) \sim \frac{2}{T_2}, \qquad (T_2 \to 0), \tag{8.2.2.11}$$

$$T_{10}(T_2) \sim \frac{4}{\sqrt{T_2}}, \qquad (T_2 \to \infty). \tag{8.2.2.12}$$

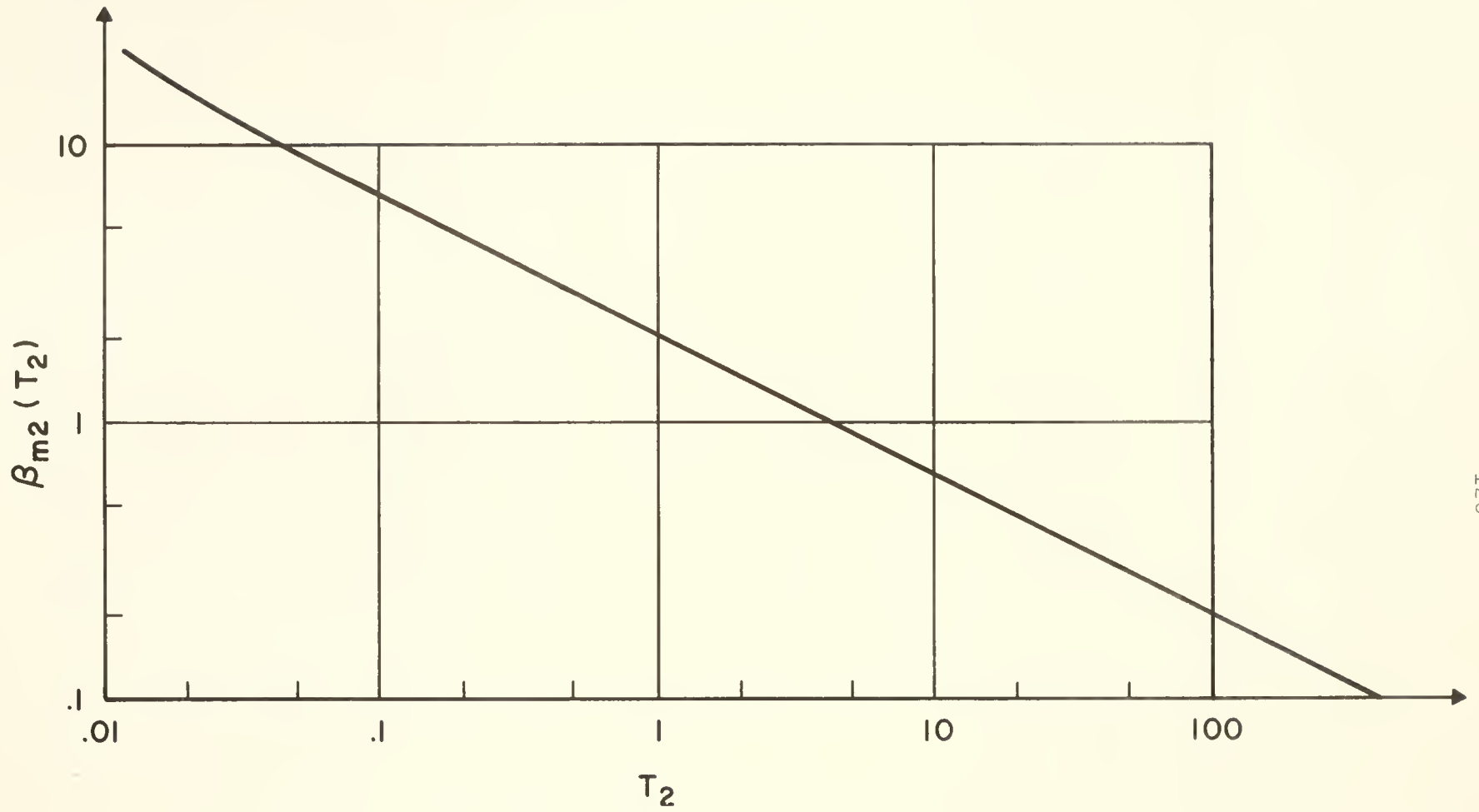

FIGURE 8.2.2.2

The functions β_{m1} of Lemma 8.2.2.2 and $\bar{T}_2$ of Lemma 8.2.2.3 appear in the following lemma.

Lemma 8.2.2.5

Let T_{20}: $R^+ \to R^+$ be defined by

$$T_{20} = \bar{T}_2 \circ \beta_{m1}.$$

Then $T_{20}'(T_1) < 0$ for $T_1 > 0$.

Proof:

Let $T_1 > 0$. Then

$$T_{20}'(T_1) = \bar{T}_2'(\beta_{m1}(T_1)) \cdot \beta_{m1}'(T_1).$$

The first factor is negative by Lemma 8.2.2.3, and the second factor is positive by Lemma 8.2.2.2. QED

Asymptotic expressions for the function T_{20} are found from expressions (8.2.2.5)-(8.2.2.8):

$$T_{20}(T_1) \sim \frac{16}{T_1^2}, \qquad (T_1 \to 0), \tag{8.2.2.13}$$

$$T_{20}(T_1) \sim \frac{2}{T_1}, \qquad (T_1 \to \infty). \tag{8.2.2.14}$$

A plot of the function T_{20} is given in Figure 8.2.2.3.

For each $T_1 > 0$, consider the function $\hat{T}_2(T_1, \cdot)$ defined on the interval $0 < \beta < T_1$ by the formula

$$\hat{T}_2(T_1, \beta) = \frac{2}{T_1 - \beta} \frac{\beta}{\bar{\psi}(\beta)}.$$

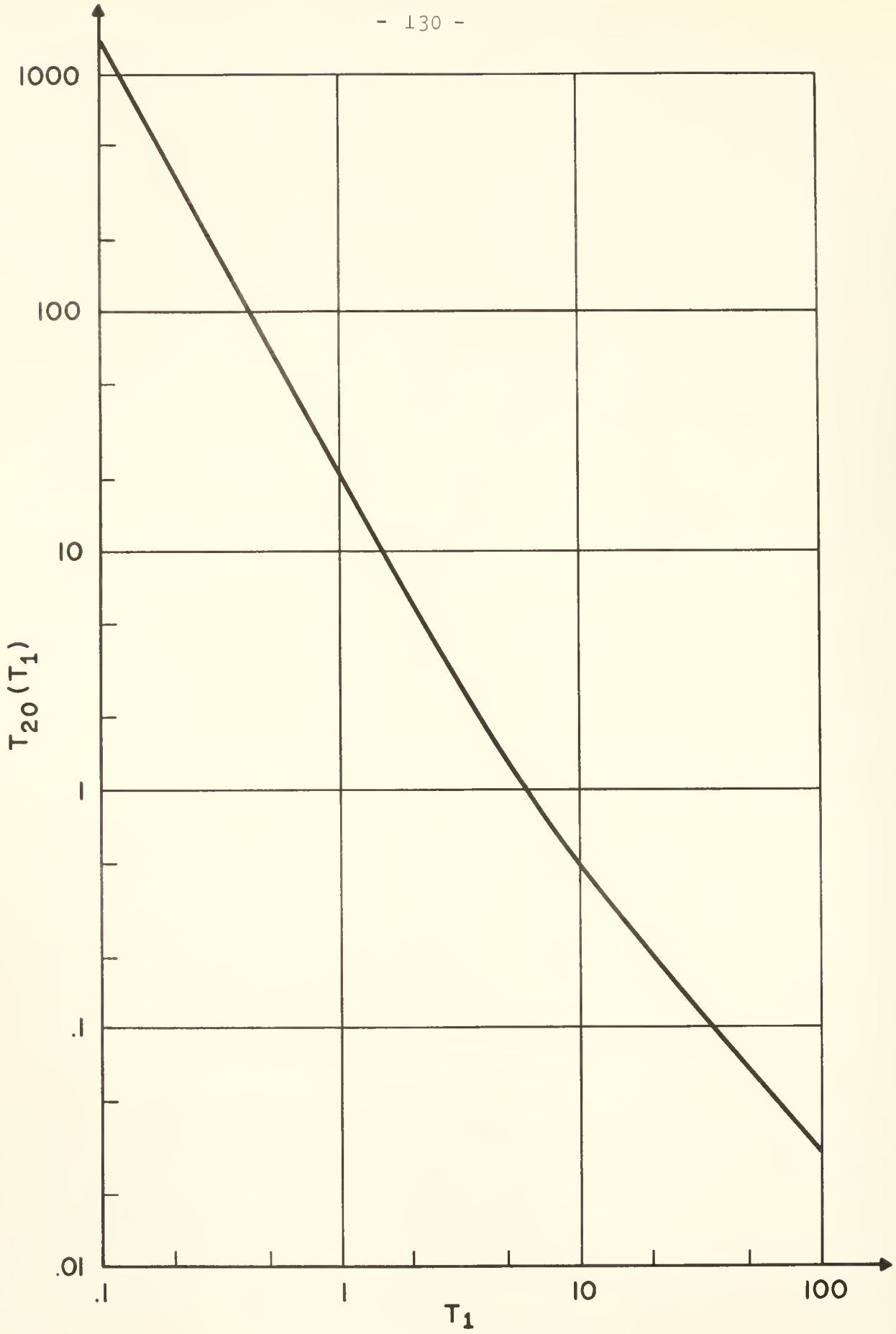

FIGURE 8.2.2.3

Let $\hat{T}_{21}(T_1,\cdot)$ and $\hat{T}_{22}(T_1,\cdot)$ denote, respectively, the restrictions of this function to the intervals $0 < \beta < \beta_{m1}(T_1)$ and $\beta_{m1}(T_1) < \beta < T_1$. We recall from Lemma 8.2.2.2 that $0 < \beta_{m1}(T_1) < \frac{1}{2} T_1$.

Lemma 8.2.2.6

The functions $\hat{T}_{2i}(T_1,\cdot)$, $i = 1,2$, take their values in the semi-infinite interval $(T_{20}(T_1),\infty)$. They are bijective and hence their inverses, denoted, respectively, by $\beta_i(T_1,\cdot)$, $i = 1,2$, exist. The derivatives of the inverses satisfy the relations

$$\frac{\partial \beta_i(T_1,T_2)}{\partial T_2} = \frac{(T_1-\beta)\beta\big(f(\beta) - 1\big)}{T_2\big(\beta f(\beta) - T_1\big)}\Bigg|_{\beta=\beta_i(T_1,T_2)}$$

and

$$(-1)^i \frac{\partial \beta_i(T_1,T_2)}{\partial T_2} > 0, \quad i = 1,2.$$

Proof:

We show first that the function $\hat{T}_2(T_1,\cdot)$ has a unique global minimum at $\beta = \beta_{m1}(T_1)$, and that its minimum value is $T_{20}(T_1)$. To this end we consider the derivative

$$\frac{\partial \hat{T}_2(T_1,\beta)}{\partial \beta}$$

$$= \frac{2}{(T_1-\beta)^2[\psi(\beta)]^2}\,[T_1\psi(\beta)-\beta(T_1-\beta)\psi'(\beta)].$$

By using (8.2.2.1) and (8.2.2.2), we get

$$\frac{\partial \hat{T}_2(T_1, \beta)}{\partial \beta}$$

$$= \frac{2\beta e^{-\beta} I_1(\beta)}{(T_1-\beta)^2[\psi(\beta)]^2} [\beta f(\beta)-T_1].$$

The derivative has the same sign as the function $\beta f(\beta) - T_1$. By Lemma 8.2.2.2, the equation $\beta f(\beta) - T_1 = 0$ has a unique solution $\beta_{m1}(T_1)$. By the same lemma, $0 < \beta_{m1}(T_1) < \frac{1}{2} T_1$, so the point where the derivative equals zero belongs to the domain of definition of the function $\hat{T}_2(T_1,\cdot)$. From Lemma 8.2.2.2 we also find that $\beta f(\beta) - T_1$ is an increasing function of β. Hence the derivative $\frac{\partial \hat{T}_2(T_1, \beta)}{\partial \beta}$ is negative for $0 < \beta < \beta_{m1}(T_1)$ and positive for $\beta_{m1}(T_1) < \beta < T_1$. We conclude that the function $\hat{T}_2(T_1,\cdot)$ has a unique global minimum at $\beta = \beta_{m1}(T_1)$. Its minimum value is $\hat{T}_2(T_1,\beta_{m1}(T_1)) = T_{20}(T_1)$. From Lemma 3.4.1(v) we find

$$\hat{T}_2(T_1, \beta) \sim \frac{4}{T_1\beta}, \qquad \beta \to 0.$$

Therefore,

$$\lim_{\beta \downarrow 0} \hat{T}_2(T_1, \beta) = \infty.$$

Also,

$$\lim_{\beta \to \infty} \hat{T}_2(T_1, \beta) = \infty.$$

We conclude that the range of each of the functions $\hat{T}_{2i}(T_1, \cdot)$, $i = 1,2$, is $(T_{20}(T_1), \infty)$.

In summary, we see that the function $\hat{T}_{21}(T_1,\cdot)$ maps the interval $(0, \beta_{m1}(T_1))$ onto $(T_{20}(T_1), \infty)$ in a strictly decreasing way, while $\hat{T}_{22}(T_1,\cdot)$ maps the interval $(\beta_{m1}(T_1),T_1)$ onto $(T_{20}(T_1), \infty)$ in a strictly increasing way.

In order to prove the expressions for the derivatives, we use the formula for the derivative of an inverse:

$$\frac{\partial \beta_i(T_1,T_2)}{\partial T_2} = \left. \frac{1}{\frac{\partial \hat{T}_{2i}(T_1, \beta)}{\partial \beta}} \right|_{\beta=\beta_i(T_1,T_2)} .$$

Substituting from above, we get

$$\frac{\partial \beta_i(T_1,T_2)}{\partial T_2} = \left. \frac{(T_1-\beta)^2[\psi(\beta)]^2}{2\beta e^{-\beta} I_1(\beta)[\beta f(\beta)-T_1]} \right|_{\beta=\beta_i(T_1,T_2)} .$$

By now noting that

$$T_2 = \left. \frac{2}{T_1-\beta} \quad \frac{\beta}{\psi(\beta)} \right|_{\beta=\beta_i(T_1,T_2)}$$

and applying (8.2.2.1) we get the expression stated in the lemma. QED

For each $T_2 > 0$, consider the function $\hat{T}_1(\cdot, T_2)$ defined for all positive values of β by the formula

$$\hat{T}_1(\beta, T_2) = \beta + \frac{2}{T_2} \frac{\beta}{\psi(\beta)}.$$

Let $\hat{T}_{11}(\cdot, T_2)$ and $\hat{T}_{12}(\cdot, T_2)$ denote, respectively, the restrictions of this function to the intervals $0 < \beta < \beta_{m2}(T_2)$ and $\beta_{m2}(T_2) < \beta$.

Lemma 8.2.2.7

The functions $\hat{T}_{1i}(\cdot, T_2)$, $i = 1,2$, take their values in the semi-infinite interval $(T_{10}(T_2), \infty)$. They are bijective and hence their inverses, denoted, respectively, by $\bar{\beta}_i(\cdot, T_2)$, $i = 1,2$, exist. The derivatives of the inverses satisfy the relations

$$\frac{\partial \bar{\beta}_i(T_1, T_2)}{\partial T_1} = \left. \frac{\beta\left(f(\beta) - 1\right)}{\beta f(\beta) - T_1} \right|_{\beta = \bar{\beta}_i(T_1, T_2)}$$

and

$$(-1)^i \frac{\partial \bar{\beta}_i(T_1, T_2)}{\partial T_1} > 0, \quad i = 1,2.$$

Proof:

We show first that the function $\hat{T}_1(\cdot, T_2)$ has a unique global minimum at $\beta = \beta_{m2}(T_2)$, and that its minimum value is $T_{10}(T_2)$. To this end we consider the derivative

$$\frac{\partial \hat{T}_1(\beta, T_2)}{\partial \beta} = 1 + \frac{2}{T_2} \frac{\psi(\beta) - \beta\psi'(\beta)}{[\psi(\beta)]^2}.$$

By using (8.2.2.1) and (8.2.2.2), we get

$$\frac{\partial \hat{T}_1(\beta, T_2)}{\partial \beta}$$

$$= 1 - \frac{2}{T_2} \quad \frac{1}{f(\beta) - 1} \quad \frac{1}{\psi(\beta)}.$$

This derivative equals zero if and only if

$$T_2 = \frac{2}{f(\beta) - 1} \quad \frac{1}{\psi(\beta)}.$$

By Lemma 8.2.2.3, this equation for β has a unique solution $\beta_{m2}(T_2) > 0$. It follows from the same lemma that $\frac{\partial \hat{T}_1(\beta, T_2)}{\partial \beta}$ is a strictly increasing function of β. Hence the derivative is negative for $0 < \beta < \beta_{m2}(T_2)$ and positive for $\beta_{m2}(T_2) < \beta$. We conclude that the function $\hat{T}_1(\cdot, T_2)$ has a unique global minimum at $\beta = \beta_{m2}(T_2)$. Its minimum value is

$$\left[\beta + \frac{2}{T_2} \frac{\beta}{\psi(\beta)}\right]\Bigg|_{\beta=\beta_{m2}(T_2)}.$$

But now $T_2 = \frac{2}{f(\beta) - 1} \quad \frac{1}{\psi(\beta)} \Big|_{\beta=\beta_{m2}(T_2)}$.

Hence the minimum value is

$$\beta f(\beta) \Big|_{\beta=\beta_{m2}(T_2)} = T_{10}(T_2).$$

From Lemma 3.4.1(v) we find

$$\lim_{\beta \downarrow 0} \hat{T}_1(\beta, T_2) = \infty.$$

Similarly, Lemma 3.4.1(vi) gives

$$\lim_{\beta \to \infty} \hat{T}_1(\beta, T_2) = \infty.$$

We conclude that the range of each of the functions $\hat{T}_{1i}(\cdot, T_2)$, $i = 1,2$, is $(T_{10}(T_2), \infty)$.

In summary, we see that the function $\hat{T}_{11}(\cdot, T_2)$ maps the interval $(0, \beta_{m2}(T_2))$ onto $(T_{10}(T_2), \infty)$ in a strictly decreasing way, while $\hat{T}_{12}(\cdot, T_2)$ maps the semi-infinite interval $(\beta_{m2}(T_2), \infty)$ onto $(T_{10}(T_2), \infty)$ in a strictly increasing way.

In order to prove the expressions for the derivatives, we use the formula for the derivative of an inverse:

$$\frac{\partial \bar{\beta}_i(T_1, T_2)}{\partial T_1} = \left. \frac{1}{\frac{\partial \hat{T}_{1i}(\beta, T_2)}{\partial \beta}} \right|_{\beta = \bar{\beta}_i(T_1, T_2)}.$$

Substituting from above, we get

$$\frac{\partial \bar{\beta}_i(T_1, T_2)}{\partial T_1} = \left. \frac{1}{1 - \frac{2}{T_2} \frac{1}{f(\beta) - 1} \frac{1}{\psi(\beta)}} \right|_{\beta = \bar{\beta}_i(T_1, T_2)}.$$

We now note that

$$T_1 = \left[\beta + \frac{2}{T_2}\frac{\beta}{\psi(\beta)}\right]\Bigg|_{\beta=\bar{\beta}_i(T_1,T_2)} .$$

Substitution and algebraic simplification now leads to the expression stated in the lemma. QED

Several of the functions that have been discussed in this section are interrelated. Some details are given in the following lemma:

Lemma 8.2.2.8

1) $T_{10} \circ T_{20} = T_{20} \circ T_{10} = I$,

2) $\beta_{m1} = \beta_{m2} \circ T_{20}$,

3) $\beta_{m2} = \beta_{m1} \circ T_{10}$,

4) $\{(T_1, T_2): T_2 > 0, T_1 > T_{10}(T_2)\}$

$= \{(T_1, T_2): T_1 > 0, T_2 > T_{20}(T_1)\}$,

5) $T_2 > T_{20}(T_1)$ implies $\beta_{m2}(T_2) < \beta_{m1}(T_1)$,

6) $\beta_i(T_1, T_2) = \bar{\beta}_i(T_1, T_2)$, $i = 1,2$ (note that $\beta_i(T_1, T_2)$ and $\bar{\beta}_i(T_1, T_2)$ are defined only for T_1, T_2 such that $T_2 > T_{20}(T_1)$),

7) $T_2 > T_{20}(T_1)$ implies

$0 < \beta_1(T_1, T_2) < \beta_{m2}(T_2) < \beta_{m1}(T_1) < \beta_2(T_1, T_2) < T_1$.

Proof:

By definition we have

$$\bar{T}_1 \circ \beta_{m1} = \beta_{m1} \circ \bar{T}_1 = I,$$

$$\bar{T}_2 \circ \beta_{m2} = \beta_{m2} \circ \bar{T}_2 = I,$$

$$T_{10} = \bar{T}_1 \circ \beta_{m2},$$

$$T_{20} = \bar{T}_2 \circ \beta_{m1}.$$

Hence

$$T_{10} \circ T_{20} = \bar{T}_1 \circ \beta_{m2} \circ \bar{T}_2 \circ \beta_{m1} = I$$

and

$$T_{20} \circ T_{10} = \bar{T}_2 \circ \beta_{m1} \circ \bar{T}_1 \circ \beta_{m2} = I,$$

which proves the first assertion. Furthermore,

$$\beta_{m2} \circ T_{20} = \beta_{m2} \circ \bar{T}_2 \circ \beta_{m1} = \beta_{m1}$$

and

$$\beta_{m1} \circ T_{10} = \beta_{m1} \circ \bar{T}_1 \circ \beta_{m2} = \beta_{m2},$$

which proves assertions 2) and 3).

Next, to prove 4), let $T_1 > T_{10}(T_2)$. Then, since T_{20} is a decreasing function, $T_{20}(T_1) < T_{20}(T_{10}(T_2)) = T_2$. Also, $T_2 > T_{20}(T_1)$ implies $T_{10}(T_2) < T_{10}(T_{20}(T_1)) = T_1$, since the function T_{10} is decreasing. Hence 4) holds.

Now let $T_2 > T_{20}(T_1)$. Since the function β_{m2} is decreasing, we get $\beta_{m2}(T_2) < \beta_{m2}(T_{20}(T_1)) = \beta_{m1}(T_1)$, which proves 5).

In order to prove 6), note first that the functions $\beta_i(T_1,\cdot)$ are defined when their argument T_2 satisfies the inequality $T_2 > T_{20}(T_1)$, while the functions $\bar{\beta}_i(\cdot,T_2)$ are defined when their argument T_1 satisfies the inequality $T_1 > T_{10}(T_2)$. By 4) above these defining conditions are equivalent.

Now, by Lemma 8.2.2.6, $\beta_1(T_1, T_2)$ is the unique solution of

$$T_2 = \frac{2}{T_1 - \beta} \frac{\beta}{\psi(\beta)}$$

that satisfies

$$0 < \beta < \beta_{m1}(T_1),$$

while, by Lemma 8.2.2.7, $\bar{\beta}_1(T_1, T_2)$ is the unique solution of

$$T_1 = \beta + \frac{2}{T_2} \frac{\beta}{\psi(\beta)}$$

which satisfies $0 < \beta < \beta_{m2}(T_2)$. Since the two equations are the same, and the first existence interval contains the second one, we conclude

$$\beta_1(T_1, T_2) = \bar{\beta}_1(T_1, T_2).$$

Similarly, $\beta_2(T_1, T_2)$ is the unique solution of

$$T_2 = \frac{2}{T_1 - \beta} \frac{\beta}{\psi(\beta)}$$

that satisfies $\beta_{m1}(T_1) < \beta < T_1$, while $\bar{\beta}_2(T_1, T_2)$ is the unique solution of

$$T_1 = \beta + \frac{2}{T_2} \frac{\beta}{\psi(\beta)}$$

that satisifes $\beta_{m2}(T_2) < \beta < \infty$. Again, the two equations are the same. Furthermore, the second existence interval contains the first one by 5). Hence

$$\beta_2(T_1, T_2) = \bar{\beta}_2(T_1, T_2).$$

Because of this relation, no confusion should arise by omitting the bar over the second function.

Relation 7) follows from the above arguments. QED

We now introduce the notation

$$0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{8.2.2.15}$$

$$w_m = \begin{pmatrix} 0 \\ \beta_{m1}(T_1) \\ \frac{1}{T_1} \beta_{m1}(T_1) \end{pmatrix}, \tag{8.2.2.16}$$

$$w_i = \begin{pmatrix} 0 \\ \beta_i(T_1, T_2) \\ \frac{1}{T_1} \beta_i(T_1, T_2) \end{pmatrix}, \quad i = 1, 2. \tag{8.2.2.17}$$

Results concerning the critical points of Problem (8.2) are contained in the following theorem.

Theorem 8.2.2.1

1) Assume $T_1 < T_{20}(T_1)$. Then Problem (8.2) has 0 as its only critical point.

2) Assume $T_1 = T_{20}(T_1)$. Then Problem (8.2) has two distinct critical points: 0 and w_m.

3) Assume $T_1 > T_{20}(T_1)$. Then Problem (8.2) has three distinct critical points: 0, w_1, and w_2.

Proof:

Let $\overset{\circ}{w} = \begin{pmatrix} \overset{\circ}{\alpha} \\ \overset{\circ}{\beta} \\ \overset{\circ}{y} \end{pmatrix}$ denote a critical point of the system $w' = h(w)$ of Section 8.2. Then $\overset{\circ}{w}$ satisfies the system of equations $h(\overset{\circ}{w}) = 0$, or

$$\begin{cases} \overset{\circ}{\alpha} = 0, \\ \overset{\circ}{y} = \frac{1}{T_1}\overset{\circ}{\beta}, \\ \overset{\circ}{y} = \dfrac{\frac{1}{2} T_2 \psi(\overset{\circ}{\beta})}{1 + \frac{1}{2} T_2 \psi(\overset{\circ}{\beta})} . \end{cases}$$

Here, we have made use of the fact that $F(0, \overset{\circ}{\beta}) = 0$, which follows from Lemma 3.6.1(i) and (8.2.5). Hence every critical point $\overset{\circ}{w}$ has the form

$$\overset{\circ}{w} = \begin{pmatrix} 0 \\ \overset{\circ}{\beta} \\ \frac{1}{T_1}\overset{\circ}{\beta} \end{pmatrix},$$

where $\overset{\circ}{\beta}$ satisfies

$$\frac{1}{T_1}\overset{\circ}{\beta} = \frac{\frac{1}{2} T_2 \psi(\overset{\circ}{\beta})}{1 + \frac{1}{2} T_2 \psi(\overset{\circ}{\beta})} . \qquad (8.2.2.18)$$

Since $\psi(0) = 0$, $\overset{\circ}{\beta} = 0$ is a solution of this equation. Also, since the right-hand side is less than one for all $\overset{\circ}{\beta} > 0$, any positive solution of (8.2.2.18) satisfies $0 < \overset{\circ}{\beta} < T_1$. For such solutions, (8.2.2.18) is equivalent to

$$T_2 = \frac{2}{T_1 - \overset{\circ}{\beta}} \frac{\overset{\circ}{\beta}}{\psi(\overset{\circ}{\beta})} . \qquad (8.2.2.19)$$

Now the right-hand side of (8.2.2.19) defines a function of T_1 and $\overset{\circ}{\beta}$. For fixed $T_1 > 0$ we consider the resulting function of $\overset{\circ}{\beta}$ for $0 < \overset{\circ}{\beta} < T_1$. In the proof of Lemma 8.2.2.6 we denoted this function by $\hat{T}_2(T_1,\cdot)$ and found that it has a minimum value of $T_{20}(T_1)$ at $\overset{\circ}{\beta} = \beta_{m1}(T_1)$. Therefore, if $T_2 < T_{20}(T_1)$, the left-hand side of (8.2.2.19) is less than the right-hand side for all values of $\overset{\circ}{\beta}$ considered. Hence (8.2.2.19) has no solution. Furthermore, if $T_2 = T_{20}(T_1)$, (8.2.2.19) is

satisfied for $\overset{\circ}{\beta} = \beta_{m1}(T_1)$, with $0 < \beta_{m1}(T_1) < \frac{1}{2} T_1$. Finally, if $T_2 > T_{20}(T_1)$, (8.2.2.19) has two roots $\beta_1(T_1, T_2)$ and $\beta_2(T_1, T_2)$, where

$$0 < \beta_1(T_1, T_2) < \beta_{m1}(T_1) < \beta_2(T_1, T_2) < T_1. \qquad \text{QED}$$

The theorem just proved allows us to evaluate the number of critical points and their coordinates for any given positive values of T_1 and T_2. We note that the critical points depend on these two parameters only, although the system $w' = h(w)$ contains the two additional parameters μ_1 and μ_2.

8.2.3 Local Stability of the Critical Points

In this subsection we use Proposition 5.4 to study local stability properties of the critical points of Problem (8.2).

Theorem 8.2.3.1

(i) The critical point 0 is asymptotically stable for all positive values of T_1 and T_2.

(ii) Assume $T_2 > T_{20}(T_1)$. Then the critical point w_1 is unstable while the critical point w_2 is asymptotically stable.

Proof: We write the system $w' = h(w)$ in the form

$$w' = Aw + b(w),$$

where

$$A = \begin{pmatrix} -\mu_1 & 0 & 0 \\ 0 & -\mu_1 & T_1\mu_1 \\ 0 & 0 & -\mu_2 \end{pmatrix}$$

and

$$b(w) = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} T_2 \mu_2 [\psi(\beta)+F(\alpha,\beta)](1-y) \end{pmatrix}.$$

Now, $\psi(\beta) = 0(\beta^2)$ as $\beta \to 0$ by Lemma 3.4.1(v) and $F(\alpha,\beta) = 0(\alpha^2) + 0(\alpha\beta)$ as $\alpha \to 0$, $\beta \to 0$, by Lemma 3.6.1(vi) and (8.2.5). Hence $|b(w)| = o(|w|)$ as $w \to 0$, so Proposition 5.4 applies. The eigenvalues of A are

$$\begin{cases} -\mu_1, \\ -\mu_1, \\ -\mu_2. \end{cases}$$

We conclude that the critical point 0 is asymptotically stable for all positive values of T_1 and T_2.

In order to investigate the stability of the critical points w_1 and w_2 for $T_2 > T_{20}(T_1)$, we denote any one of these critical points by

$$\overset{\circ}{w} = \begin{pmatrix} 0 \\ \circ \\ \beta \\ \circ \\ y \end{pmatrix},$$

and introduce

$$\tilde{w} = \begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \\ \tilde{y} \end{pmatrix}$$

by $\tilde{w} = w - \overset{\circ}{w}$.

Then

$$\psi(\beta) = \psi(\overset{\circ}{\beta}) + \tilde{\beta}\, \psi'(\overset{\circ}{\beta}) + \psi_R(\tilde{\beta})$$

and

$$F(\alpha,\beta) = F(0,\overset{\circ}{\beta}) + \tilde{\alpha}\, F_\alpha(0,\overset{\circ}{\beta}) + \tilde{\beta}\, F_\beta(0,\overset{\circ}{\beta}) + F_R(\tilde{\alpha},\tilde{\beta}).$$

By now applying Lemma 3.6.1(i) and (vii) and relation (8.2.5) we get $F(0,\overset{\circ}{\beta}) = F_\beta(0,\overset{\circ}{\beta}) = 0$ and $F_\alpha(0,\overset{\circ}{\beta}) = \frac{n}{N_1}\, \psi'(\overset{\circ}{\beta})$. Hence $F(\alpha,\beta) = \tilde{\alpha}\, \frac{n}{N_1}\, \psi'(\overset{\circ}{\beta}) + F_R(\tilde{\alpha},\tilde{\beta})$. The remainder functions ψ_R and F_R can be written

$$\psi_R(\tilde{\beta}) = 0(\tilde{\beta}^2) \text{ as } \tilde{\beta}\to 0$$

and

$$F_R(\tilde{\alpha},\tilde{\beta}) = 0(\tilde{\alpha}^2) + 0(\tilde{\alpha}\,\tilde{\beta}) + 0(\tilde{\beta}^2) \text{ as } \tilde{\alpha}\to 0,\ \tilde{\beta}\to 0.$$

Hence

$$b(w) = b(\overset{\circ}{w}) + B\tilde{w} + \tilde{b}(\tilde{w}),$$

where

$$B = \frac{1}{2} T_2 \mu_2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{n}{N_1} \psi'(\overset{\circ}{\beta})(1-\overset{\circ}{y}) & \psi'(\overset{\circ}{\beta})(1-\overset{\circ}{y}) & -\psi(\overset{\circ}{\beta}) \end{pmatrix}$$

and $|\tilde{b}(\tilde{w})| = o(|\tilde{w}|)$ as $\tilde{w} \to 0$.

Therefore,

$$h(w) = Aw + b(w) = A\tilde{w} + A\overset{\circ}{w} + b(\overset{\circ}{w}) + B\tilde{w} + \tilde{b}(\tilde{w})$$

or

$$h(w) = \tilde{A}\,\tilde{w} + \tilde{b}(\tilde{w})$$

where

$$\tilde{A} = A + B.$$

Proposition 5.4 now applies. The matrix $\tilde{A}$ has three eigenvalues. One of them is $-\mu_1$, while the other two satisfy the equation

$$\lambda^2 + a\lambda + b = 0,$$

where

$$a = \mu_1 + \mu_2 + \frac{1}{2} T_2 \mu_2 \psi(\overset{\circ}{\beta})$$

and

$$b = \mu_1\mu_2 + \frac{1}{2} T_2 \mu_1\mu_2 \psi(\overset{\circ}{\beta}) - \frac{1}{2} T_1 T_2 \mu_1\mu_2 \psi'(\overset{\circ}{\beta})(1-\overset{\circ}{y}).$$

The solutions of the equation in λ can be written

$$\lambda = -\frac{1}{2}\, a \pm \frac{1}{2}\, \sqrt{a^2 - 4b}.$$

Now

$$a^2 - 4b = (\mu_1 - \mu_2 - \tfrac{1}{2}\, T_2\mu_2\, \psi(\overset{\circ}{\beta}))^2 + 2\, T_1T_2\, \mu_1\mu_2\, \psi'(\overset{\circ}{\beta})(1-\overset{\circ}{y}) \geq 0.$$

Hence all eigenvalues of $\tilde{A}$ are real.

Since $a > 0$, we conclude that one of the roots of the λ-equation is always negative. The other of the two roots has the same sign as $-b$, or the same sign as

$$f_1(\overset{\circ}{\beta}) = -\frac{2b}{\mu_1\mu_2T_2e^{-\overset{\circ}{\beta}}I_1(\overset{\circ}{\beta})}$$

$$= -\frac{1}{e^{-\overset{\circ}{\beta}}I_1(\overset{\circ}{\beta})}\left[\frac{2}{T_2} + \psi(\overset{\circ}{\beta}) - \psi'(\overset{\circ}{\beta})(T_1-\overset{\circ}{\beta})\right].$$

Now $\overset{\circ}{\beta}$ satisfies relation (8.2.2.19). Hence

$$f_1(\overset{\circ}{\beta}) = -\frac{1}{e^{-\overset{\circ}{\beta}}I_1(\overset{\circ}{\beta})}\left[\frac{T_1\psi(\overset{\circ}{\beta})}{\overset{\circ}{\beta}} - \psi'(\overset{\circ}{\beta})(T_1-\overset{\circ}{\beta})\right].$$

By using relations (8.2.2.1) and (8.2.2.2) we get

$$f_1(\overset{\circ}{\beta}) = T_1 - \overset{\circ}{\beta}f(\overset{\circ}{\beta}).$$

From Lemma 8.2.2.2, $\beta f(\beta)$ is a strictly increasing function. From the definition of the function β_{m1} in the same lemma we have that $\beta_{m1}(T_1)f(\beta_{m1}(T_1)) = T_1$. Hence we conclude that

$$f_1(\overset{\circ}{\beta}) > 0 \text{ for } 0 < \overset{\circ}{\beta} < \beta_{m1}(T_1)$$

and

$$f_1(\overset{\circ}{\beta}) < 0 \text{ for } \beta_{m1}(T_1) < \overset{\circ}{\beta} < T_1.$$

Both of the roots of the λ-equation above are therefore negative for $\beta_{m1}(T_1) < \overset{\circ}{\beta} < T_1$, while one of them is positive for $0 < \overset{\circ}{\beta} < \beta_{m1}(T_1)$. Now the former of these two inequalities is satisfied by $\beta_2(T_1, T_2)$, while the latter one is satisfied by $\beta_1(T_1, T_2)$. Hence statement (ii) follows.

QED

8.2.4 Stability in the Large of the Critical Points

In this subsection we complement the results of the previous subsection by proving some stability properties in the large of the critical points of Problem (8.2). In addition to notation that has been used earlier in this chapter, we introduce the following:

$$B^{\circ} = \{(\alpha,\beta,y): y = \frac{1}{T_1}\beta\}, \tag{8.2.4.1}$$

$$Y^{\circ} = \left\{ (\alpha,\beta,y): y = \frac{\frac{T_2}{2}[\psi(\beta)+F(\alpha,\beta)]}{1 + \frac{T_2}{2}[\psi(\beta)+F(\alpha,\beta)]} \right\}, \tag{8.2.4.2}$$

$$B^+ = \{(\alpha,\beta,y): \quad y > \frac{1}{T_1}\beta\},$$

$$B^- = \{(\alpha,\beta,y): \quad y < \frac{1}{T_1}\beta\},$$

$$Y^+ = \left\{(\alpha,\beta,y): \quad y < \frac{\frac{T_2}{2}[\psi(\beta)+F(\alpha,\beta)]}{1+\frac{T_2}{2}[\psi(\beta)+F(\alpha,\beta)]}\right\},$$

$$Y^- = \left\{(\alpha,\beta,y): \quad y > \frac{\frac{T_2}{2}[\psi(\beta)+F(\alpha,\beta)]}{1+\frac{T_2}{2}[\psi(\beta)+F(\alpha,\beta)]}\right\},$$

$$M_1 = B^+ \cap Y^+,$$
$$M_2 = B^- \cap Y^-,$$
$$M_3 = B^+ \cap Y^-,$$
$$M_4 = B^- \cap Y^+.$$

We use the rectangular parallelepiped C of (8.2.1.2) and put $N_i = M_i \cap C$, $i = 1,\ldots, 4$. Figures 8.2.4.1 - 8.2.4.3 show the intersections between these sets and the plane $\alpha = 0$ in the three cases $T_2 < T_{20}(T_1)$, $T_2 = T_{20}(T_1)$, and $T_2 > T_{20}(T_1)$, respectively.

Lemma 8.2.4.1

The sets $\bar{N}_2$ and $\bar{N}_1 \cup \bar{N}_2 \cup \bar{N}_3$ are invariant with respect to the system $w' = h(w)$ of Section 8.2.

Proof:

Both the sets mentioned in the lemma are bounded and closed. It suffices therefore, by Proposition 5.2, to show that the boundary points of each set are ingress points of the corresponding set. But from Lemma 8.2.1.2

we find that all boundary points of C are ingress points of C. By an application of Proposition 5.1 (iii) we therefore conclude that it suffices to show that those boundary points of the sets $\bar{M}_1$ and $\bar{M}_1 \cup \bar{M}_2 \cup \bar{M}_3$ that belong to C are ingress points of $\bar{M}_1$ and $\bar{M}_1 \cup \bar{M}_2 \cup \bar{M}_3$, respectively. Now every point that belongs to the boundaries of these sets belongs to $(B^\circ \cup Y^\circ) \cap C$. Thus we consider the initial-value problem

$$\begin{cases} w' = h(w), \\ w(0) = \bar{w}, \end{cases} \tag{II}$$

where $\bar{w} \in (B^\circ \cup Y^\circ) \cap C$. By allowing $\bar{w}$ to range over this entire set, all boundary points of $\bar{M}_2$ and $\bar{M}_1 \cup \bar{M}_2 \cup \bar{M}_3$ that belong to C are covered. Now the set $(B^\circ \cup Y^\circ) \cap C$ can be partitioned into the union of five disjoint sets as follows:

$$(B^\circ \cup Y^\circ) \cap C = \bigcup_{i=1}^{5} B_i,$$

where

$$B_1 = B^\circ \cap Y^+ \cap C,$$
$$B_2 = B^\circ \cap Y^- \cap C,$$
$$B_3 = B^+ \cap Y^\circ \cap C,$$
$$B_4 = B^- \cap Y^\circ \cap C,$$

and

$$B_5 = B^\circ \cap Y^\circ \cap C.$$

Consider now $\bar{w} \in B_1$. Then $\bar{w}$ belongs to the boundaries of those portions of the sets B^+ and B^- that are contained in Y^+, i.e. $\bar{w}$ belongs to the boundaries of

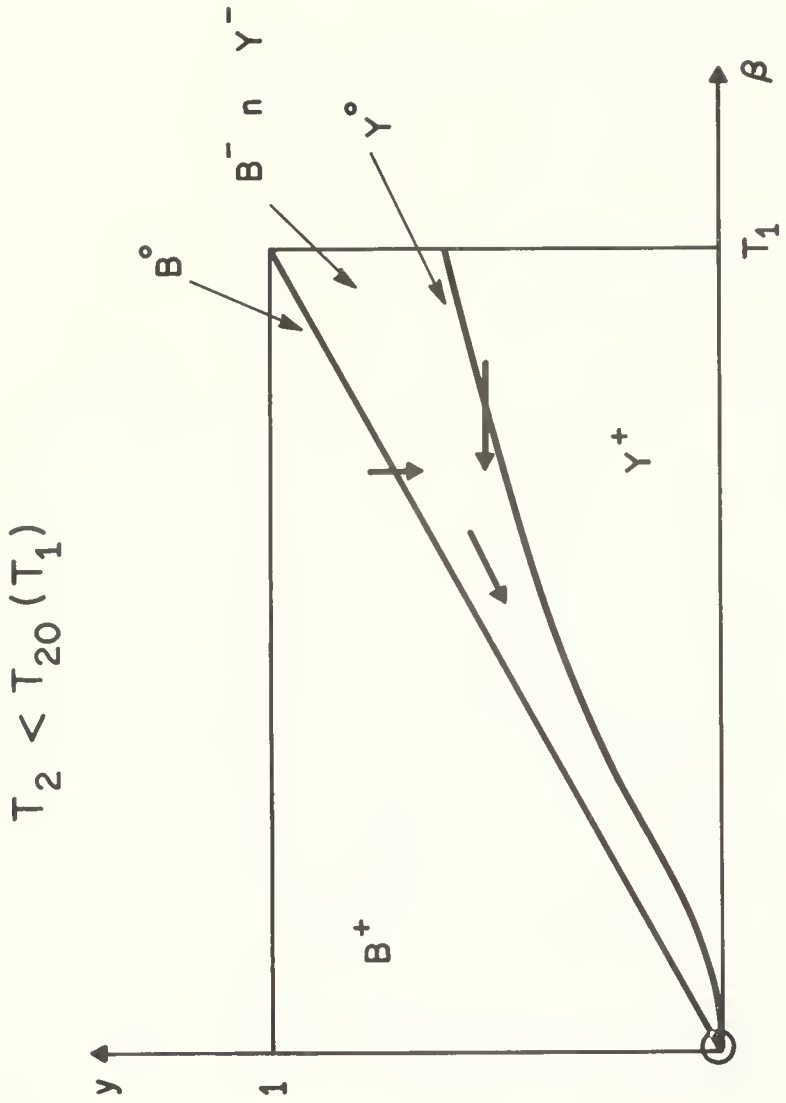

FIGURE 8.2.4.1

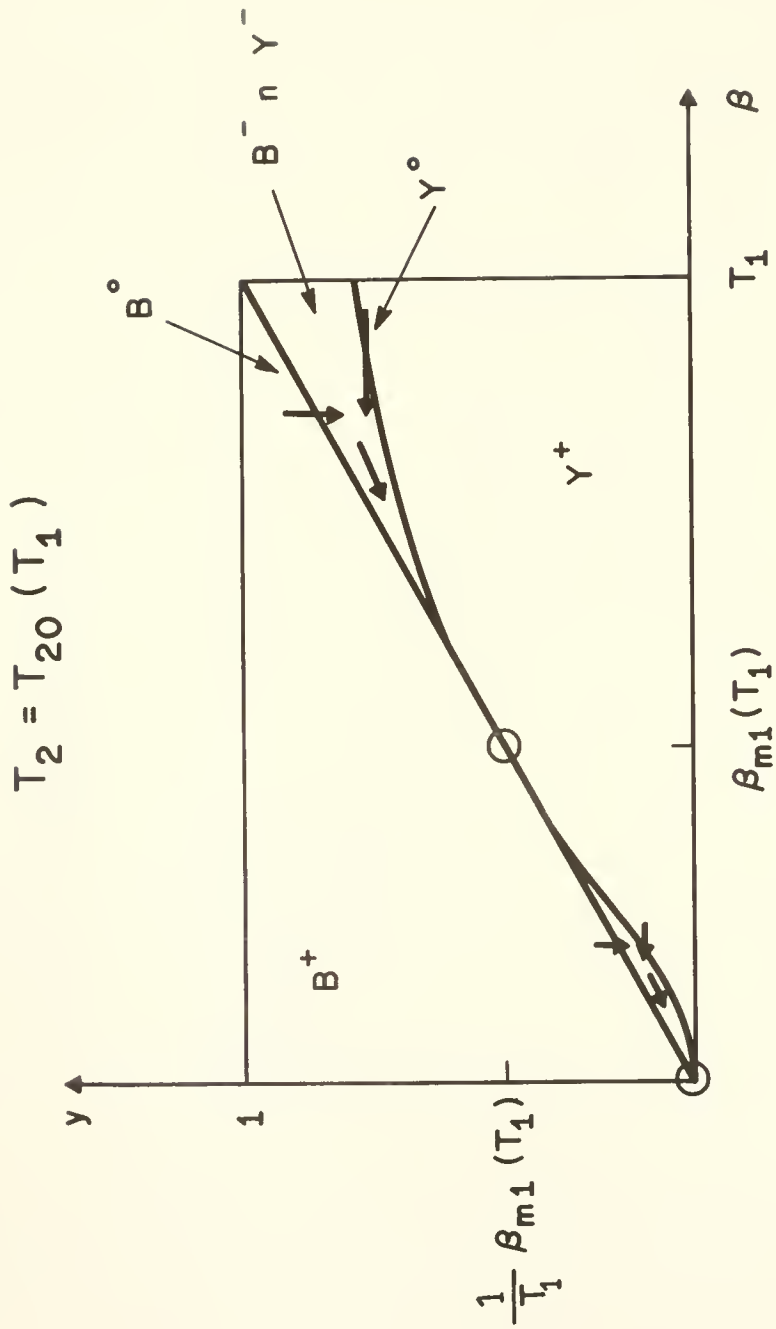
$T_2 = T_{20}(T_1)$
B°
$B^- \cap Y^-$
Y°
B^+
Y^+
y
1
$\frac{1}{T_1}\beta_{m1}(T_1)$
T_1
β
$\beta_{m1}(T_1)$

FIGURE 8.2.4.2

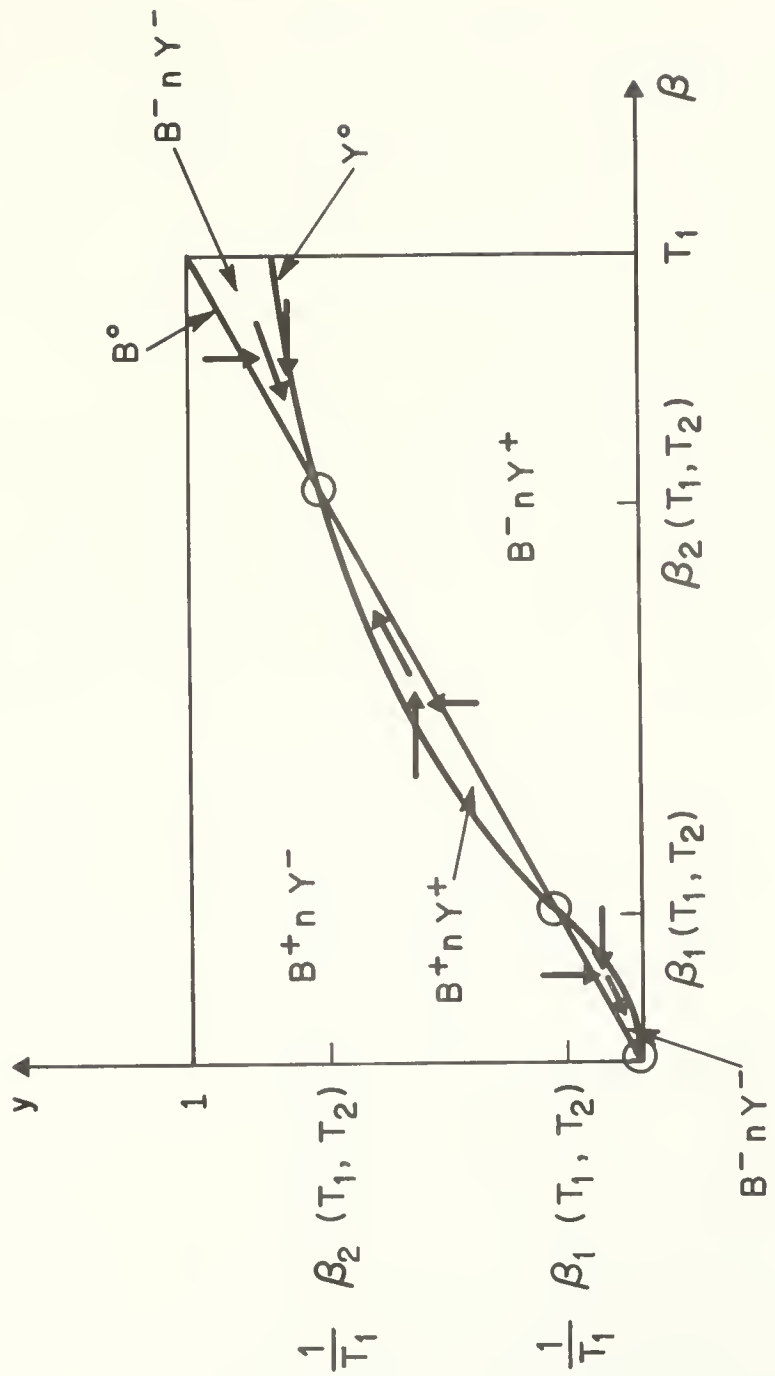
$T_2 > T_{20}$ (T_1)
$B^-{}_n Y^-$
Y°
B°
$B^-{}_n Y^+$
$B^+{}_n Y^-$
$B^+{}_n Y^+$
$B^-{}_n Y^-$
y
1
$\frac{1}{T_1}$ β_2 (T_1, T_2)
$\frac{1}{T_1}$ β_1 (T_1, T_2)
$\beta_1 (T_1, T_2)$
$\beta_2 (T_1, T_2)$
T_1
β

FIGURE 8.2.4.3

the sets $M_1 = B^+ \cap Y^+$ and $M_4 = B^- \cap Y^+$. We note also that $\bar{w}$ does not belong to the boundaries of M_2 or M_3, since $\bar{w} \in Y^+$. In a similar way we find that if $\bar{w} \in B_2$, then $\bar{w}$ is a boundary point of M_2 and M_3 but not of M_1 or M_4, if $\bar{w} \in B_3$, then $\bar{w}$ is a boundary point of M_1 and M_3 but not of M_2 or M_4, and if $\bar{w} \in B_4$, then $\bar{w}$ is a boundary point of M_2 and M_4 but not M_1 or M_3.

In proving the lemma, we need be concerned only with boundary points of the sets $\bar{M}_2$ and $\bar{M}_1 \cup \bar{M}_2 \cup \bar{M}_3$. From the preceding paragraph we find: If $\bar{w} \in B_1$, it suffices to show that $\bar{w}$ is an ingress point of $\bar{M}_1$. If $\bar{w} \in B_2$, it suffices to show that $\bar{w}$ is an ingress point of $\bar{M}_2$. If $\bar{w} \in B_4$, it suffices to show that $\bar{w}$ is an ingress point of $\bar{M}_2$. The case $\bar{w} \in B_3$ need not be considered, since $\bar{w}$ is then not a boundary point of either $\bar{M}_2$ or $\bar{M}_1 \cup \bar{M}_2 \cup \bar{M}_3$. The case $\bar{w} \in B_5$ requires some special considerations and is deferred for later treatment.

Now define a function B by

$$B(\alpha,\beta,y) = y - \frac{\beta}{T_1} .$$

Then B° is the plane defined by $B(\alpha,\beta,y) = 0$. The vector

$$N_B = (0, -\frac{1}{T_1}, 1)$$

is normal to B° and is directed into the set B^+. Problem (II) with $\bar{w} \in B^\circ$ has a solution with tangent at $\bar{w}$ given by the vector

$$T(0) = (h_1(\bar{w}),\ 0,\ h_3(\bar{w})).$$

Hence

$$N_B \cdot T(0) = h_3(\bar{w}).$$

Now $h_3(\bar{w}) > 0$ for $\bar{w} \in Y^+$ and $h_3(\bar{w}) < 0$ for $\bar{w} \in Y^-$.

With $\bar{w} \in B_1 \subset B^\circ \cap Y^+$ we find by Proposition 5.3 that $\bar{w}$ is an ingress point of B^+. Proposition 5.1 (i) then shows that $\bar{w}$ is an ingress point of $M_1 = B^+ \cap Y^+$.

With $\bar{w} \in B_2 \subset B^\circ \cap Y^-$ we find by the observation following Proposition 5.3 that $\bar{w}$ is an ingress point of B^-. Proposition 5.1 (i) then shows that $\bar{w}$ is an ingress point of $M_2 = B^- \cap Y^-$.

Next define a function $\hat{Y}$ by

$$\hat{Y}(\alpha,\beta,y) = \frac{\frac{1}{2}\, T_2[\psi(\beta)+F(\alpha,\beta)]}{1 + \frac{1}{2}\, T_2[\psi(\beta)+F(\alpha,\beta)]} - y.$$

Then Y° is the surface defined by $\hat{Y}(\alpha,\beta,y) = 0$. Now

$$\hat{Y}_\alpha(\alpha,\beta,y) = \frac{T_2 F_\alpha(\alpha,\beta)}{2[1 + \frac{1}{2}\, T_2(\psi(\beta)+F(\alpha,\beta))]^2},$$

$$\hat{Y}_{\beta}(\alpha,\beta,y) = \frac{T_2(\psi'(\beta)+F_{\beta}(\alpha,\beta))}{2[1 + \frac{1}{2} T_2(\psi(\beta)+F(\alpha,\beta))]^2} ,$$

and

$$\hat{Y}_{y}(\alpha,\beta,y) = - 1.$$

Using these expressions, the vector

$$N_Y = (Y_{\alpha}(\alpha,\beta,y), Y_{\beta}(\alpha,\beta,y), Y_{y}(\alpha,\beta,y))$$

is normal to the surface Y° at the point (α,β,y), and it is directed into the set Y^{+}. Problem (II) with $\bar{w} \in Y^{\circ}$ has a solution with tangent at $\bar{w}$ given by the vector $T(0)$ $= (h_1(\bar{w}), h_2(\bar{w}), 0)$.

Hence

$$N_Y \cdot T(0) = \frac{T_2}{2[1 + \frac{1}{2} T_2(\psi(\bar{\beta})+F(\bar{\alpha},\bar{\beta}))]^2}$$

$$\cdot [h_1(\bar{w})F_{\alpha}(\bar{\alpha},\bar{\beta})+h_2(\bar{w})(\psi'(\bar{\beta})+F_{\beta}(\bar{\alpha},\bar{\beta}))].$$

The sign of this inner product is determined by the second expression within brackets.

Let $\bar{w} \in B_4 = B^- \cap Y^\circ \cap C$. Then $h_2(\bar{w}) < 0$. Since $\bar{\beta} > 0$ we have from Lemma 3.4.1(ii) that $\psi'(\bar{\beta}) > 0$. Also, Lemma 3.6.5 and (8.2.5) give $F_\beta(\bar{\alpha},\bar{\beta}) \geq 0$, while Lemma 3.6.9 and (8.2.5) give $F_\alpha(\bar{\alpha},\bar{\beta}) \geq 0$. Furthermore, $h_1(\bar{w}) = -\mu_1\bar{\alpha} \leq 0$. Therefore $N_Y \cdot T(0) < 0$. From the observation following Proposition 5.3 we deduce that $\bar{w}$ is an ingress point of Y^-. Proposition 5.1(i) then shows that $\bar{w}$ is an ingress point of $M_2 = B^- \cap Y^-$.

We finally consider the case $\bar{w} \in B_5 = B^\circ \cap Y^\circ \cap C$. We then have $N_B \cdot T(0) = 0$, while $N_Y \cdot T(0)$ has the same sign as $-\bar{\alpha}\, F_\alpha(\bar{\alpha},\bar{\beta})$. Five different mutually exclusive subcases are considered:

1) $\bar{\alpha} = 0$;

2) $\bar{\alpha} > 0$, $f_k \vee m_k = 0$, $k = 1,\ldots,N_1$;

3) $\bar{\alpha} > 0$, $f_k \vee m_k > 0$ for some $k = 1,\ldots,N_1$,
$f_k \wedge m_k = 0$, $k = 1,\ldots,N_1$, $\bar{\beta} = 0$;

4) $\bar{\alpha} > 0$, $f_k \vee m_k > 0$ for some $k = 1,\ldots,N_1$,
$f_k \wedge m_k = 0$, $k = 1,\ldots,N_1$, $\bar{\beta} > 0$;

5) $\bar{\alpha} > 0$, $f_k \wedge m_k > 0$ for some $k = 1,\ldots,N_1$.

In case 1) we have $\bar{\alpha} = 0$. Hence the point $\bar{w}$ is a critical point. The solution of Problem (III) is then $w(t) = \bar{w}$, $t \geq 0$. The point $\bar{w}$ is hence an ingress point of the closure of each set of which it is a boundary point.

In case 2) we have $f_k = m_k = 0$ for $k = 1,\ldots,N_1$. From Lemma 3.6.1(ii) and (8.2.5) follows that $F(\alpha,\beta) = 0$. The solution of Problem (II) is then

$$w(t) = \begin{pmatrix} \bar{\alpha}\, e^{-\mu_1 t} \\ \bar{\beta} \\ \bar{y} \end{pmatrix}, \quad t \geq 0.$$

Suppose first that $\bar{w} \in \partial M_2$. In order to prove that $\bar{w}$ is an ingress point of $\bar{M}_2$, it suffices to show that $w(t) \in \partial M_2$ for all $t \geq 0$. Let $N_r(w)$ denote an open sphere with center at w and radius r. Fix $t > 0$ and $\varepsilon > 0$. It then suffices to show that $N_\varepsilon(w(t))$ contains a point in M_2 and a point in M_2^c (the complement of M_2). Now $\bar{w} \in B^\circ$ implies $\bar{y} = \frac{1}{T_1}\bar{\beta}$. Hence $w(t) \notin M_2$. Next $\bar{w} \in \partial M_2$ implies that there exists $w^* = \begin{pmatrix} \alpha^* \\ \beta^* \\ y^* \end{pmatrix}$ in $N_\varepsilon(\bar{w})$ such that $w^* \in M_2$. But then

$$\frac{\frac{1}{2} T_2 \psi(\beta^*)}{1 + \frac{1}{2} T_2 \psi(\beta^*)} < y^* < \frac{1}{T_1}\beta^*.$$

Consider now the point $w^*(t) = \begin{pmatrix} \alpha^* e^{-\mu_1 t} \\ \beta^* \\ y^* \end{pmatrix}$. The above inequalities are satisfied, so $w^*(t) \in M_2$. Also, $w^*(t) \in N_\varepsilon(w(t))$. We conclude that $\bar{w}$ is an ingress point of $\bar{M}_2$. In a similar way we show that any $\bar{w}$ on the boundary of $\bar{M}_1 \cup \bar{M}_2 \cup \bar{M}_3$ is an ingress point of $\bar{M}_1 \cup \bar{M}_2 \cup \bar{M}_3$.

In case 3) we find from Lemma 3.6.1(iv) and (8.2.5) that $F(\bar{\alpha},0) = 0$. The facts $\bar{\beta} = 0$ and $\bar{w} \in B^\circ$ imply $\bar{y} = 0$. The solution of Problem (II) is found to be

$$w(t) = \begin{pmatrix} \bar{\alpha}\, e^{-\mu_1 t} \\ 0 \\ 0 \end{pmatrix}, \quad t \geq 0.$$

For brevity write $\bar{M}_1 \cup \bar{M}_2 \cup \bar{M}_3 = M$ and assume $\bar{w} \in \partial M$. In order to prove that $\bar{w}$ is an ingress point of M, it suffices to show that $w(t) \in \partial M$ for all $t \geq 0$. Fix $t > 0$ and $\varepsilon > 0$. It then suffices to show that $N_\varepsilon(w(t))$ contains a point in $\overset{\circ}{M}$ and a point in $\overset{\circ}{M}{}^c$. From the definitions of the sets M_i we have

$$\overset{\circ}{M} = \left\{(\alpha,\beta,y): \quad y > \min\left[\frac{1}{T_1}\beta,\ \gamma(\alpha,\beta)\right]\right\}$$

where

$$\gamma(\alpha,\beta) = \frac{\frac{1}{2}T_2[\psi(\beta) + F(\alpha,\beta)]}{1 + \frac{1}{2}T_2[\psi(\beta) + F(\alpha,\beta)]}.$$

We note that the solution $w(t)$ given above belongs to $B^\circ \cap Y^\circ$. Indeed, all points $(\alpha,0,0)$ belong to B°. They also belong to Y° since $F(\alpha,0) = 0$ and $\psi(0) = 0$ imply that $\gamma(\alpha,0) = 0$.

Hence $w(t) \in \overset{\circ}{M}^c$. Next $\bar{w} \in \partial M$ implies that there exists $w^* = \begin{pmatrix} \alpha^* \\ \beta^* \\ y^* \end{pmatrix}$ in $N_\varepsilon(\bar{w})$ such that $w^* \in \overset{\circ}{M}$. Consider the point $w^*(t) = \begin{pmatrix} \alpha^* e^{-\mu_1 t} \\ \beta^* \\ y^* \end{pmatrix}$. This point belongs to the ε-neighborhood $N_\varepsilon(w(t))$ of $w(t)$. We complete the proof of case 3) by showing that $w^*(t) \in \overset{\circ}{M}$. Since $w^* \in \overset{\circ}{M}$ we have $y^* > \min \left[\frac{1}{T_1} \beta^*, \gamma(\alpha^*, \beta^*) \right]$. Now $\frac{\partial}{\partial \alpha} \gamma(\alpha,\beta) \geq 0$ since $F_\alpha(\alpha,\beta) \geq 0$, which follows from Lemma 3.6.9 and (8.2.5).

Hence $\gamma(\alpha^*,\beta^*) \geq \gamma(\alpha^* e^{-\mu_1 t}, \beta^*)$, $t \geq 0$ and $y^* > \min \left[\frac{1}{T_1} \beta^*, \gamma(\alpha^* e^{-\mu_1 t}, \beta^*) \right]$. We conclude that $w^*(t) \in \overset{\circ}{M}$. In a similar manner we show that any $\bar{w}$ on the boundary of M_2 is an ingress point of $\bar{M}_2$.

Cases 4) and 5) are treated simultaneously. The three main facts to be used for both cases are that $\bar{\alpha} > 0$, $\bar{\beta} > 0$, and $f_k \vee m_k > 0$ for some $k = 1,\ldots,N_1$. The fact that $\bar{\beta} > 0$ in case 5) is seen as follows: Lemma 3.6.1(v) and (8.2.5) show that $F(\bar{\alpha},0) > 0$. The point $\bar{w} = \begin{pmatrix} \bar{\alpha} \\ 0 \\ 0 \end{pmatrix}$ does therefore not belong to Y°. We note that $\bar{w}$ is a boundary point of the sets Y^- and B^-. From Lemma 3.6.9 and (8.2.5) we have $F_\alpha(\bar{\alpha},\bar{\beta}) > 0$. Hence $N_Y \cdot T(0)$ is strictly negative, which shows that $\bar{w}$ is an ingress point of Y^-. In order to show that $\bar{w}$ is an ingress point of B^-, it suffices to show that there exists $\varepsilon > 0$ such that $y(t) < \frac{1}{T_1} \beta(t)$ for all t in the interval $0 < t < \varepsilon$. Suppose that no such ε exists.

Then there exists a sequence $\{t_n\}$ of positive numbers with the properties that $t_n \downarrow 0$ as $n \to \infty$ and

$$y(t_n) \geq \frac{1}{T_1} \beta(t_n), \quad n = 1,2,\ldots .$$

Hence

$$\frac{y(t_n) - y(0)}{t_n^2} \geq \frac{1}{T_1} \frac{\beta(t_n) - \beta(0)}{t_n^2}, \quad n = 1,2,\ldots . \tag{8.2.4.3}$$

But now

$$\beta'(0) = 0,$$

$$y'(0) = 0,$$

$$\beta''(0) = 0,$$

and

$$y''(0) = -\frac{1}{2} T_2 \mu_1 \mu_2 \bar{\alpha}(1-\bar{y}) F_\alpha(\bar{\alpha},\bar{\beta}) < 0.$$

The function y'' is continuous since F_α and F_β are. From Taylor's theorem we conclude that the left-hand side of (8.2.4.3) approaches $\frac{1}{2} y''(0) < 0$ as $n \to \infty$, while the right-hand side approaches $\frac{1}{2T_1} \beta''(0) = 0$. This contradiction to (8.2.4.3) proves the assertion that $\bar{w}$ is an ingress point of B^-. We have thus shown that $\bar{w}$ is an ingress point of Y^- and of B^-. We conclude that $\bar{w}$ is a boundary point of $M_2 = B^- \cap Y^-$ and an ingress point of M_2.

We note also that $\bar{w}$ is an ingress point of $\bar{M}_1 \cup \bar{M}_2 \cup \bar{M}_3$ whenever $\bar{w}$ is a boundary point of the same set.

This completes the proof of the assertion that the sets $\bar{N}_2$ and $\bar{N}_1 \cup \bar{N}_2 \cup \bar{N}_3$ are invariant with respect to the system $w' = h(w)$. QED

We now proceed to the main result of the subsection.

Theorem 8.2.4.1 Consider the initial-value problem

$$\begin{cases} w' = h(w), \\ w(0) = w_o, \end{cases}$$

of Section 8.2

I. If $T_2 < T_{20}$ (T_1), then the solution $w(t)$ approaches the asymptotically stable critical point 0 as $t \to \infty$.

II. If $T_2 = T_{20}(T_1)$, then the critical point w_m defined in (8.2.2.16) is unstable, and the solution $w(t)$ approaches either the asymptotically stable critical point 0 or the unstable critical point w_m as $t \to \infty$.

III. If $T_2 > T_{20}$ (T_1), then the solution $w(t)$ either approaches one of the asymptotically stable critical points 0 or w_2, or it approaches the unstable critical point w_1 as $t \to \infty$; here, w_1 and w_2 are given in (8.2.2.17).

Proof: In Theorems 8.2.2.1 and 8.2.3.1 we showed that:

I. If $T_2 < T_{20}$ (T_1), then 0 is the only critical point of the problem and it is asymptotically stable.

II. If $T_2 = T_{20}(T_1)$, then the problem has two critical points: 0 and w_m, where the point 0 is asymptotically stable.

III. If $T_2 > T_{20}(T_1)$, then the problem has three critical points: 0, w_1, and w_2. Among these points, 0 and w_2 are asymptotically stable, while w_1 is unstable.

In proving the theorem, it therefore suffices to show that

1) every solution approaches a critical point as $t \to \infty$, and

2) if $T_2 = T_{20}(T_1)$, then the critical point w_m is unstable.

We show first that every solution of Problem (8.2) approaches a critical point as $t \to \infty$.

Whatever the values of f_k, m_k, and y_o are, $k = 1,\ldots,N_1$, the initial point w_o always belongs to the closure of the set B^+. Hence we have $w_o \in \bar{N}_1 \cup \bar{N}_3$. Now the larger set $\bar{N}_1 \cup \bar{N}_2 \cup \bar{N}_3$ is invariant by Lemma 8.2.4.1. Hence the solution remains in this larger set for all $t \geq 0$. Two cases are distinguished. In the first case, the solution enters the set $\bar{N}_2$ at some finite time $t_o \geq 0$, and in the second case it does not. In the former case we make use of the monotonicity and invariance of the set $\bar{N}_2$. An application of Proposition 5.6(i) then shows that the solution approaches a critical point as $t \to \infty$.

In the latter case, the solution remains in the set $\bar{N}_1 \cup \bar{N}_3 = \overline{B^+} \cap C$ for all $t \geq 0$. But $\alpha(t) \to 0$ as $t \to \infty$,

and the limit β_∞ of $\beta(t)$ as $t \to \infty$ exists by boundedness and monotonicity. The third equation in $w' = h(w)$ can be written

$$y'(t) + p(t)y(t) = q(t),$$

where

$$q(t) = \frac{1}{2} T_2\mu_2[\psi(\beta(t)) + F(\alpha(t),\beta(t))]$$

and $$p(t) = q(t) + \mu_2.$$

The solution of this equation is

$$y(t) = e^{-\int_0^t p(s)ds}\left[\int_0^t q(u)e^{\int_0^u p(s)ds}\,du + y(0)\right].$$

As $t \to \infty$ we find by an application of l'Hospital's rule that $y(t)$ approaches the value

$$y_\infty = \lim_{t\to\infty} \frac{q(t)}{p(t)} = \frac{\frac{1}{2} T_2\psi(\beta_\infty)}{1 + \frac{1}{2} T_2\psi(\beta_\infty)}$$

since $F(0,\beta_\infty) = 0$. The second equation in $w' = h(w)$ is

$$\beta'(t) + \mu_1\beta(t) = T_1\mu_1 y(t).$$

Its solution is

$$\beta(t) = e^{-\mu_1 t}\left[\int_0^t T_1\mu_1 y(s)\, e^{\mu_1 s}\, ds + \beta(0)\right].$$

By l'Hospital's rule we find that $\beta(t)$ approaches $T_1 y_\infty$ as $t \to \infty$.

Hence we have

$$\beta_\infty = T_1 y_\infty.$$

We have thus shown that the solution approaches $\begin{pmatrix} 0 \\ \beta_\infty \\ y_\infty \end{pmatrix}$ as $t \to \infty$. This is a critical point of the system $w' = h(w)$.

To conclude the proof of the theorem, we now show that if $T_2 = T_{20}(T_1)$, then the critical point w_m is unstable.

Let $\varepsilon > 0$ be chosen such that $\varepsilon < \beta_{m1}(T_1)$. Then take $0 < \varepsilon_B < \dfrac{\varepsilon}{\sqrt{1 + \frac{4}{T_1^2}}}$ and choose $\bar{\beta}$ in the interval

$$\beta_{m1}(T_1) - \varepsilon_B < \bar{\beta} < \beta_{m1}(T_1).$$

By the proof of Lemma 8.2.2.6, the function $\hat{T}_2(T_1, \cdot)$ has a unique global minimum at $\beta = \beta_{ml}(T_1)$, and its minimum value is $T_{20}(T_1)$. Hence $\hat{T}_2(T_1, \bar{\beta}) > T_{20}(T_1) = T_2$ or

$$\frac{2}{T_1 - \bar{\beta}} \quad \frac{\bar{\beta}}{\psi(\bar{\beta})} > T_2$$

or

$$\frac{1}{T_1} \bar{\beta} > \frac{\frac{1}{2} T_2 \psi(\bar{\beta})}{1 + \frac{1}{2} T_2 \psi(\bar{\beta})} . \tag{8.2.4.4}$$

Now take ε_Y such that

$$\frac{\varepsilon_B}{T_1} < \varepsilon_Y < \frac{2\varepsilon_B}{T_1}$$ and choose $\bar{y}$ in the interval

$$\frac{1}{T_1} \beta_{ml}(T_1) - \varepsilon_Y < \bar{y} < \frac{1}{T_1} \beta_{ml}(T_1)$$

such that

$$\frac{\frac{1}{2} T_2 \psi(\bar{\beta})}{1 + \frac{1}{2} T_2 \psi(\bar{\beta})} < \bar{y} < \frac{1}{T_1} \bar{\beta} . \tag{8.2.4.5}$$

This can be done since

$$\frac{1}{T_1}\beta_{m1}(T_1) - \varepsilon_Y < \frac{1}{T_1}\left[\beta_{m1}(T_1) - \varepsilon_B\right] < \frac{1}{T_1}\bar{\beta} < \frac{1}{T_1}\beta_{m1}(T_1)$$

and (8.2.4.4) holds.

Now the point $\bar{w} = \begin{pmatrix} 0 \\ \bar{\beta} \\ \bar{y} \end{pmatrix}$ belongs to $\bar{B} \cap Y^-$ because of (8.2.4.5). Furthermore,

$$\left[\beta_{m1}(T_1) - \bar{\beta}\right]^2 + \left[\frac{1}{T_1}\beta_{m1}(T_1) - \bar{y}\right]^2$$

$$< \varepsilon_B^2 + \varepsilon_Y^2 < \varepsilon^2.$$

Let $w(t)$, $t \geq 0$, denote the solution of the initial-value problem

$$\begin{cases} w' = h(w), \\ w(0) = \bar{w}. \end{cases}$$

This solution approaches a critical point as $t \to \infty$. Now $\bar{w} \in B^- \cap Y^- \cap C = N_2$, and the set $\bar{N}_2$ is invariant. The solution therefore remains in $\bar{N}_2$ for all $t \geq 0$, and the β-coordinate is a decreasing function of t. Hence $\beta(t) \leq \bar{\beta} < \beta_{m1}(T_1)$ for all $t \geq 0$. This implies that the solution approaches 0 as $t \to \infty$. Hence the critical point w_m is unstable. QED

8.3 Existence and Uniqueness of the Solution of the Model

Theorem 8.3.1

There exists a unique solution of the model of Section 8.1.

Proof:

In order to prove uniqueness, assume that there exists at least one solution of the model of Section 8.1. From the arguments in Section 8.2 it then follows that the function y defined in (8.2.1) satisfies the initial-value problem of Section 8.2. From Theorem 8.2.1.1 we conclude that the function $y(t)$ is uniquely determined, and that it is non-negative, bounded, and continuous. For each $k = 1,2,\ldots,N_1$, consider now the system (8.1.1) of differential-difference equations with initial values for the functions $P_{k;f}(t)$. In this system, replace the expression $\sum_{i=1}^{N_2} i\Pi_i(t)$ by $N_2 y(t) = Y(t)$. The resulting system is formally identical with system (3.1.7). But the latter system has a unique solution as noted at the end of Section 3.1. Hence the functions $P_{k;f}(t)$ are uniquely determined. In a similar way we relate systems (8.1.2) and (3.1.8) to conclude that the functions $Q_{k;m}(t)$ are uniquely determined. From the results in Chapter 3 we can now evaluate what we refer to as the expectation of the number of paired female parasites. It is uniquely determined from the functions $P_{k;f}(t)$ and $Q_{k;m}(t)$ by the expression

$$\sum_{p=1}^{\infty} p \left[P_{k;p}(t) \sum_{j=0}^{\infty} Q_{k;p+j}(t) + Q_{k;p}(t) \sum_{i=1}^{\infty} P_{k;p+i}(t) \right].$$

As is done in (8.2.2), we use the notation $Z(t) = N_1 z(t)$ to denote the sum of the above expressions over all k-values from 1 to N_1. We note that the functions $Z(t)$ and $z(t)$ are uniquely determined. Furthermore, they are both non-negative, bounded, and continuous, which follows from (8.2.4). Now consider the system (8.1.3) of differential-difference equations with initial values for the functions $\Pi_i(t)$. In this system, replace the expression

$$\sum_{k=1}^{N_1} \sum_{p=1}^{\infty} p \left[P_{k;p}(t) \sum_{j=0}^{\infty} Q_{k;p+j}(t) + Q_{k;p}(t) \sum_{i=1}^{\infty} P_{k;p+i}(t) \right]$$

by $Z(t)$. The resulting system is formally identical with system (4.1.1). But the latter system has a unique solution as noted at the end of Section 4.1. Hence the functions $\Pi_i(t)$ are uniquely determined. This concludes the proof of the uniqueness statement.

In order to prove existence of a solution to the model, let $\begin{pmatrix} \alpha^*(t) \\ \beta^*(t) \\ y^*(t) \end{pmatrix}$, $t \geq 0$, denote the solution of Problem (8.2). The function $y^*(t)$ is then non-negative, bounded, and continuous by Theorem 8.2.1.1. For each $k = 1,\ldots,N_1$,

consider the system (3.1.7) with n_o replaced by n_k, $P_f(t)$ replaced by $P_{k;f}(t)$, and $N_2 y^*(t)$ serving for the function $Y(t)$. From Chapter 3 it follows that this system can be solved uniquely for the functions $P_{k;f}(t)$. In a similar way we find that the system (3.1.8) properly modified can be solved uniquely for the functions $Q_{k;m}(t)$. From these functions we get the function $Z(t)$ by relation (8.2.2). From $Z(t) = N_1 z(t)$ and (8.2.4) it follows that this function is non-negative, bounded, and continuous. Consider now the system (4.1.1) with the above function $Z(t)$ replacing the function $X(t)$. From Chapter 4 it follows that this system can be solved uniquely for the functions $\Pi_i(t)$. We claim that the resulting collection of functions $P_{k;f}(t)$, $Q_{k;m}(t)$, and $\Pi_i(t)$, with $k = 1,\ldots,N_1, f,m = 0,1,2,\ldots$, and $i = 1,2,\ldots,N_2$, constitutes a solution of the model of Section 8.1. In order to prove this claim we compare systems (3.1.7), (3.1.8) and (4.1.1) with systems (8.1.1)-(8.1.3). We find from this comparison that it suffices to show that

$$y^*(t) = \frac{1}{N_2} \sum_{i=1}^{N_2} i\Pi_i(t). \qquad (8.3.1)$$

From (4.2.5) and (8.2.4) we find the following differential equation to be satisfied by $\frac{1}{N_2} \sum_{i=1}^{N_2} i\Pi_i(t)$:

$$\frac{d}{dt}\left[\frac{1}{N_2}\sum_{i=1}^{N_2} i\Pi_i(t)\right] = \frac{1}{2}T_2\mu_2\left[\psi(\beta^*(t)) + F(\alpha^*(t),\beta^*(t))\right]$$

$$\cdot\left[1 - \frac{1}{N_2}\sum_{i=1}^{N_2} i\Pi_i(t)\right] - \mu_2\frac{1}{N_2}\sum_{i=1}^{N_2} i\Pi_i(t).$$

At t = 0 we have from (4.2.6)

$$\frac{1}{N_2}\sum_{i=1}^{N_2} i\Pi_i(0) = y_o.$$

A comparison with (8.2.6) and (8.2.7) now shows that $y^*(t)$ and $\frac{1}{N_2}\sum_{i=1}^{N_2} i\Pi_i(t)$ satisfy the same linear initial-value problem. We conclude that (8.3.1) holds, since the initial-value problem has a unique solution. QED

Chapter 9

Epidemiology of Schistosomiasis

The discussion in this chapter parallels that of Chapter 7. A comparison of the results in this chapter with the results in Chapter 7 illuminates the epidemiological differences caused by the different mating patterns of mature parasites assumed in the two chapters.

9.1 The Threshold

Before we give epidemiological interpretations of the threshold effect that exists in this case, we establish the following lemma. The notation follows that of Chapter 8.

Lemma 9.1.1

$$\frac{d}{dT_1}\left[T_1 \cdot T_{20}(T_1)\right] = -\frac{2}{[f(\beta)-1]^2\,\psi(\beta)}\Bigg|_{\beta=\beta_{m1}(T_1)}.$$

Proof:

By definition of the function T_{20} we have

$$T_1 \cdot T_{20}(T_1) = \frac{2T_1}{f(\beta)-1}\,\frac{1}{\psi(\beta)}\Bigg|_{\beta=\beta_{m1}(T_1)}.$$

Differentiation of this expression gives

$$\frac{d}{dT_1}\left[T_1 \cdot T_{20}(T_1)\right]$$

$$= \left\{\frac{2}{(f(\beta)-1)^2(\psi(\beta))^2}\left[(f(\beta)-1)\psi(\beta)\right.\right.$$

$$\left.\left. - T_1\beta_{m1}'(T_1)[f'(\beta)\psi(\beta) + (f(\beta)-1)\psi'(\beta)]\right]\right\} \Bigg|_{\beta=\beta_{m1}(T_1)}.$$

Now the definition of the function β_{m1} gives

$$\beta_{m1}'(T_1) = \frac{1}{f(\beta)+\beta f'(\beta)} \Bigg|_{\beta=\beta_{m1}(T_1)}.$$

From (8.2.2.1) and (8.2.2.2) we get

$$(f(\beta)-1)\psi'(\beta) = \frac{1}{\beta} f(\beta)\psi(\beta).$$

Hence

$$\beta_{m1}'(T_1)[f'(\beta)\psi(\beta) + (f(\beta)-1)\psi'(\beta)] \Big|_{\beta=\beta_{m1}(T_1)}$$

$$= \frac{\psi(\beta)}{\beta} \Big|_{\beta=\beta_{m1}(T_1)}.$$

The lemma now follows by algebraic simplification, using the fact that

$$[\beta f(\beta)] \Big|_{\beta=\beta_{m1}(T_1)} = T_1.$$ QED

Consider now a community where no disease exists and where some disease is introduced at time t = 0 through infected definitive or intermediate hosts. On the basis of Theorem 8.2.4.1 we conclude that a necessary condition for the establishment of a positive prevalence of disease in the community is that $T_2 \geq T_{20}$ (T_1). Statement I of the same theorem also allows us to conclude that if $T_2 < T_{20}$ (T_1), then the disease introduced at time t = 0 will ultimately die out, regardless of the amount of disease originally introduced. The amount of disease introduced is measured by the number of infected intermediate hosts and the number of mature parasites alive inside the bodies of infected definitive hosts.

In case $T_2 > T_{20}$ (T_1), we note from statement III of Theorem 8.2.4.1 that every solution approaches one of the critical points 0, w_1, and w_2 as $t \to \infty$. By an application of Proposition 5.5 we find that there exists a neighborhood of the critical point w_1 such that the set of points in that neighborhood for which the solution ultimately approaches w_1 has measure 0, while the set of points in that neighborhood for which the solution does not ultimately approach w_1 (and therefore ultimately approaches one of the other critical points 0 or w_2) is dense. On the basis of this result, we rule out the practical occurrence of any solution that ultimately approaches the critical point w_1.

In case $T_2 = T_{20}$ (T_1), we note from statement II of Theorem 8.2.4.1 that every solution approaches one of the critical points 0 and w_m. Among these, the point w_m is unstable. However, an interpretation of solutions approaching this point as being practically impossible can not be made here. The difference compared to the case with the point w_1 above is that Proposition 5.5 can not be applied here, since the largest eigenvalue of w_m is 0 and not strictly positive.

For the further discussions in this chapter of the case $T_2 > T_{20}$ (T_1) we note that it can be shown that there exist two non-empty subsets of $(2N_1 + 1)$ - dimensional space such that the solution of Problem (8.2) approaches the critical point 0 as $t \to \infty$ if the initial data given by f_k, m_k, and y_o, $k = 1,\ldots,N_1$, belong to the first of these sets, and the solution approaches the critical point w_2 as $t \to \infty$ if the initial data belong to the second of these sets. An investigation of properties of the above-mentioned subsets will be the subject of future investigations. It is anticipated that the results of such investigations will have meaningful epidemiological interpretations.

We now return to a discussion of a community where no disease exists. We consider the case $T_2 > T_{20}$ (T_1) and the long-term effects on the community of the introduction of some disease at time $t = 0$. Since both the critical points 0 and w_2 are asymptotically stable, we conclude that

if the amount of disease introduced is sufficiently small, then the disease will ultimately die out, while if the amount of disease introduced is sufficiently large, then the disease will ultimately establish itself at a positive level of endemicity.

We finally use the threshold phenomenon to discuss eradication of the disease when it is already established at a positive level of endemicity. Two conceptually different ways of eradication are at our disposal. In the first one, no change is made in biological and environmental parameters. Thus, the transmission factors T_1 and T_2 are unchanged. A sufficiently large reduction in the number of alive mature parasites and the proportion of infected intermediate hosts then leads to eradication of the disease. We shall not pursue this method of eradication further here; rather, we concentrate on a second approach which involves a decrease in one or both of the transmission factors T_1 and T_2. This second approach has the epidemiological advantage, commented on above, that the resulting endemic level of prevalence at zero is stable in the large, i.e. any amount of disease introduced into the community will ultimately die out.

In order to eradicate the disease in this manner, it is necessary and sufficient to reduce the product of the transmission factors T_1 and T_2 to a value less than or equal to $T_1 \cdot T_{20}(T_1)$. By Lemma 9.1.1, this function of T_1 is

decreasing for all positive values of T_1. By relations (8.2.2.13) and (8.2.2.14) we find that it has the following asymptotic expressions:

$$T_1 \cdot T_{20}(T_1) \sim \frac{16}{T_1}, \quad (T_1 \to 0),$$

and

$$T_1 \cdot T_{20}(T_1) \sim 2, \quad (T_1 \to \infty).$$

The first of these relations shows that $T_1 \cdot T_{20}(T_1) \to \infty$ as $T_1 \downarrow 0$. This means, in particular, that it is possible for the product T_1T_2 to be arbitrarily large in a community that cannot support a positive endemic level of prevalence, provided T_1 is small enough. Figure 9.1.1 gives a plot of $T_1 \cdot T_{20}(T_1)$ as a function of T_1.

It was noted in Section 7.1 that in order to eradicate hermaphroditic helminthiasis, a reduction in T_1 is as efficient as a reduction in T_2. A different situation prevails for schistosomiasis, where a reduction in T_1 is more efficient. To see this, assume $T_2 > T_{20}(T_1)$. Then the disease can be eradicated by either

1) keeping T_2 fixed and reducing T_1 to a value $\overset{\circ}{T}_1$ such that $T_{20}(\overset{\circ}{T}_1) = T_2$, or
2) keeping T_1 fixed and reducing T_2 to $T_{20}(T_1)$.

In the first case, eradication requires a reduction in T_1 by the factor $\frac{\overset{\circ}{T}_1}{T_1}$, while in the second case eradication requires a reduction in T_2 by the factor $\frac{T_{20}(T_1)}{T_2} = \frac{T_{20}(T_1)}{T_{20}(\overset{\circ}{T}_1)}$. By Lemma 9.1.1, the first factor is larger than the second one.

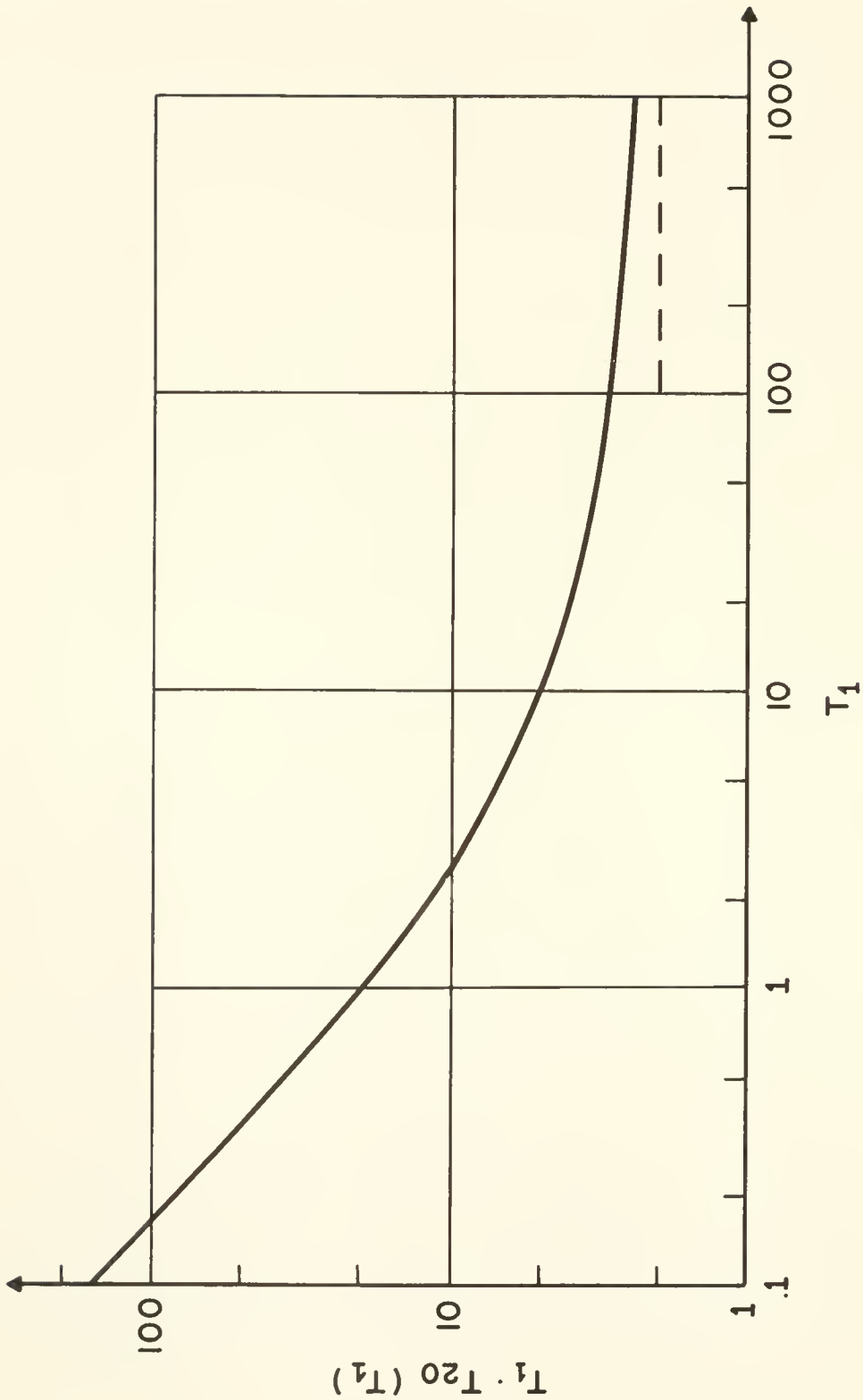

FIGURE 9.1.1

9.2 The Prevalence

In this section we study the prevalence of schistosomiasis in the case where $T_2 > T_{20}$ (T_1) and where the initial data (given by f_k, m_k, and y_o, $k = 1,\ldots,N_1$) are such that the solution of Problem (8.2) approaches w_2 as $t \to \infty$. We assume that a sufficiently long time has elapsed so that the prevalence is close to its asymptotic value. The prevalence is measured by the proportion of definitive hosts that contain at least one pair of mature and alive parasites in their bodies, i.e. by $P\{\xi_P(t) > 0\}$. From (3.5.7) we find

$$P\{\xi_P(t) > 0\} \to \left(1 - e^{-\frac{1}{2}\beta}\right)^2$$

as $t \to \infty$, where $\beta = \beta_2(T_1, T_2)$. The steady-state prevalence is thus a function of T_1 and T_2. Denoting it by Q_s (where the subscript s refers to schistosomiasis), we have

$$Q_s(T_1, T_2) = \left(1 - e^{-\frac{1}{2}\beta_2(T_1, T_2)}\right)^2, \qquad (9.2.1)$$

where the function $\beta_2 = \overline{\beta}_2$ is defined in Lemmas 8.2.2.6 and 8.2.2.7. In these lemmas it is shown that the function β_2 is increasing in both of its arguments. From expression (9.2.1) we therefore conclude that the steady-state prevalence is decreased by a decrease in either T_1 or T_2.

For fixed $T_1 > 0$ we find from Lemma 8.2.2.6 that $\beta_2(T_1, T_2) \to T_1$ as $T_2 \to \infty$. For fixed $T_2 > 0$ we find from Lemma 8.2.2.7 that $\beta_2(T_1, T_2) \sim T_1$ as $T_1 \to \infty$. Hence we find for fixed $T_1 > 0$ that $Q_s(T_1, T_2) \to \left(1 - e^{-\frac{1}{2} T_1}\right)^2$ as $T_2 \to \infty$. On the other hand, for fixed $T_2 > 0$, we get $Q_s(T_1, T_2) \to 1$ as $T_1 \to \infty$.

Figure 9.2.1 gives a plot of the steady-state prevalence Q_s as a function of T_2 for $T_1 = 4$. For comparison the figure also gives the prevalence Q_h from Section 7.2. The different asymptotes as $T_2 \to \infty$ are indicated by the plots. It is also noteworthy that when the steady-state prevalence for schistosomiasis is different from 0, then for every fixed T_1 (or T_2) it has a strictly positive lower bound. Figure 9.2.2 gives the corresponding curves for $T_2 = 4$. Here it is seen that the steady-state prevalences for both hermaphroditic helminthiasis and schistosomiasis approach the value 1 as $T_1 \to \infty$.

In order to compare efficiencies of controls of the prevalence, we compute

$$T_1 \frac{\partial Q_s(T_1, T_2)}{\partial T_1} = T_1 \frac{\partial \beta_2(T_1, T_2)}{\partial T_1} \left(e^{-\frac{1}{2}\beta} - e^{-\beta}\right) \Bigg|_{\beta=\beta_2(T_1,T_2)}$$

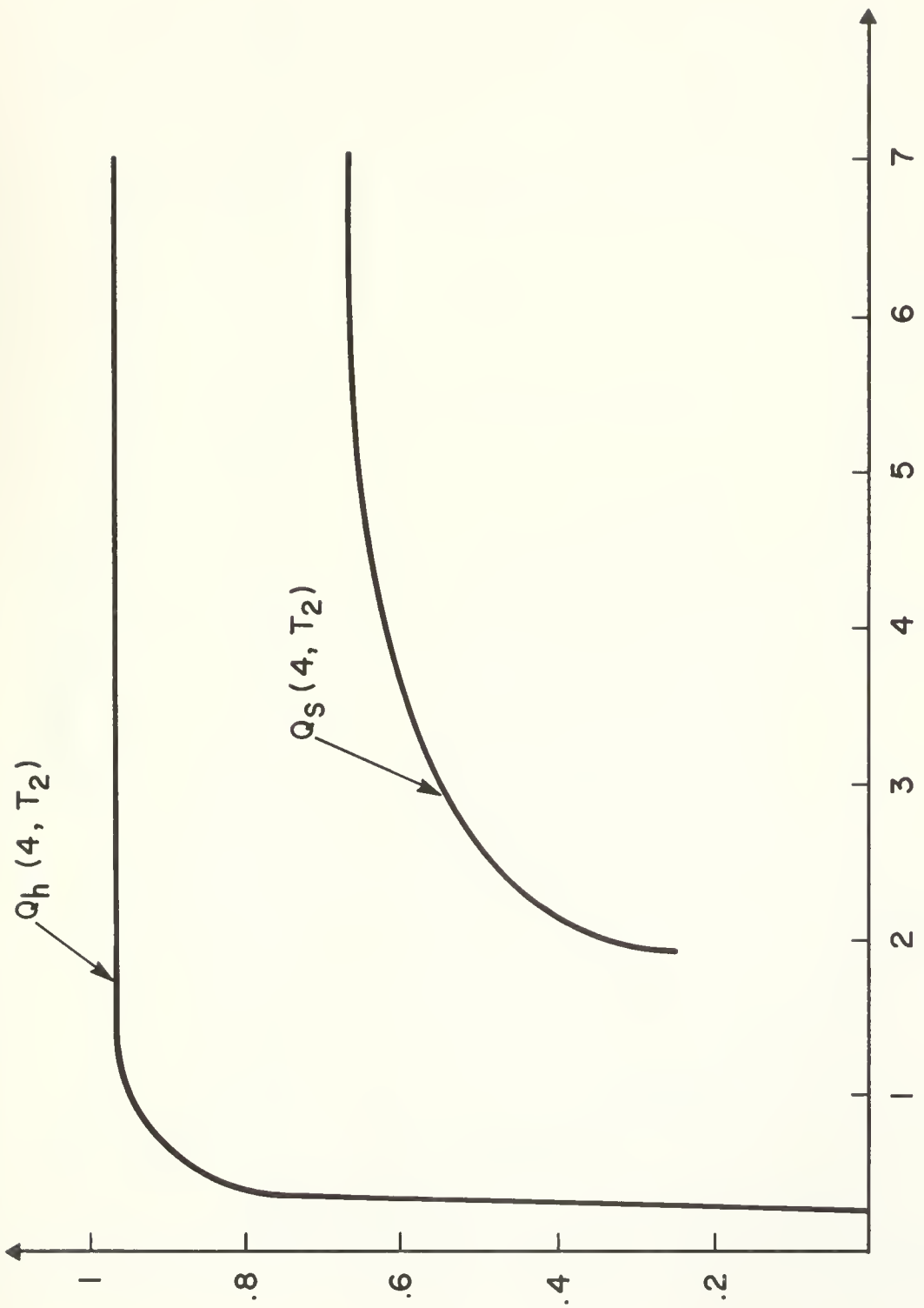

FIGURE 9.2.1

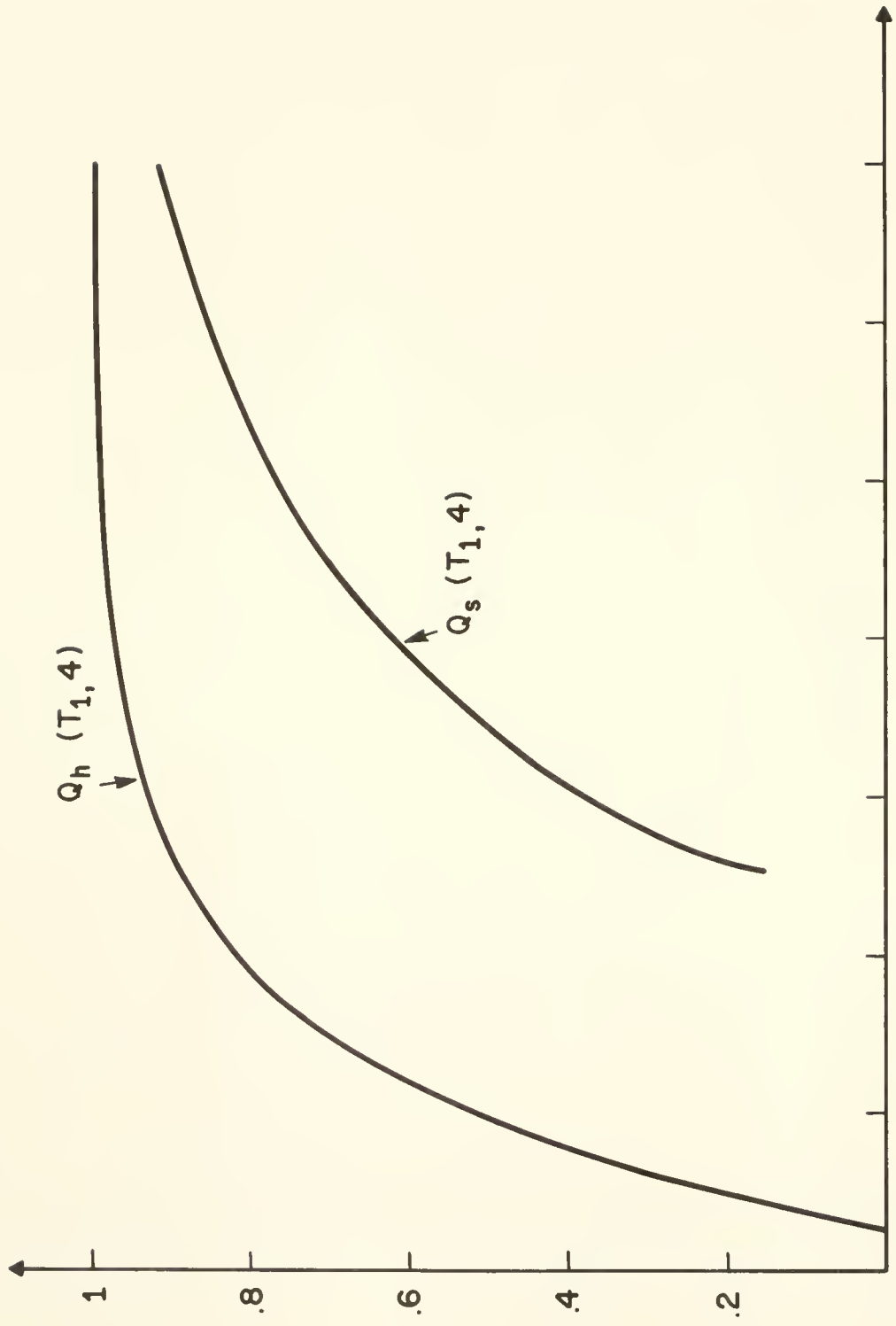

FIGURE 9.2.2

and

$$T_2 \frac{\partial Q_s(T_1, T_2)}{\partial T_2} = T_2 \frac{\partial \beta_2(T_1, T_2)}{\partial T_2} \left(e^{-\frac{1}{2}\beta} - e^{-\beta} \right) \Bigg|_{\beta=\beta_2(T_1,T_2)}$$

Using Lemmas 8.2.2.6 and 8.2.2.7, we find

$$T_1 \frac{\partial Q_s(T_1, T_2)}{\partial T_1} = \frac{T_1\beta(f(\beta)-1)\left(e^{-\frac{1}{2}\beta} - e^{-\beta} \right)}{\beta f(\beta) - T_1} \Bigg|_{\beta=\beta_2(T_1,T_2)} \tag{9.2.2}$$

and

$$T_2 \frac{\partial Q_s(T_1, T_2)}{\partial T_2} = \frac{(T_1-\beta)\beta(f(\beta)-1)\left(e^{-\frac{1}{2}\beta} - e^{-\beta} \right)}{\beta f(\beta) - T_1} \Bigg|_{\beta=\beta_2(T_1,T_2)} . \tag{9.2.3}$$

Now the right-hand side of (9.2.2) is larger than the right-hand side of (9.2.3). Hence we have

$$T_1 \frac{\partial Q_s(T_1, T_2)}{\partial T_1} > T_2 \frac{\partial Q_s(T_1, T_2)}{\partial T_2} .$$

Using an argument similar to that in Section 7.2, we conclude that the decrease in Q_s accompanying a sufficiently small decrease in T_1 is larger than the decrease in Q_s that accompanies a proportionately equal decrease in T_2. We express this by saying that control of the prevalence of schistosomiasis through a decrease of T_1 is more efficient than control through a decrease of T_2. By comparing right-hand sides of (9.2.2) and (9.2.3) it is seen that the ratio of efficiencies is appreciable if $T_2 >> T_{20}$ (T_1), since $\beta_2(T_1, T_2) \to T_1$ as $T_2 \to \infty$.

9.3 The Proportion of Intermediate Hosts that are Infected

In this section we study the proportion of intermediate hosts that are infected in the case where $T_2 > T_{20}$ (T_1) and where the initial data (given by f_k, m_k, and y_o, $k = 1,\ldots,N_1$) are such that the solution of Problem (8.2) approaches w_2 as $t \to \infty$. We assume that a sufficiently long time has elapsed so that the proportion of infected intermediate hosts is close to its asymptotic value. The proportion of intermediate hosts that are infected at time t is given by $y(t)$, $t \geq 0$. By Theorem 8.2.4.1, $y(t) \to \frac{1}{T_1} \beta_2(T_1, T_2)$ as $t \to \infty$. The steady-state proportion is thus a function of the transmission factors T_1 and T_2. Denoting this function by I_s, we have

$$I_s\ (T_1,T_2) = \frac{1}{T_1}\ \beta_2(T_1,T_2), \tag{9.3.1}$$

where the function $\beta_2 = \bar{\beta}_2$ is defined in Lemmas 8.2.2.6 and 8.2.2.7. By Lemma 8.2.2.6 we have

$$\frac{\partial \beta_2(T_1,T_2)}{\partial T_2} > 0.$$

Hence

$$\frac{\partial I_s(T_1,T_2)}{\partial T_2} > 0,$$

which shows that a decrease in T_2 leads to a decrease in I_s. Next, using Lemma 8.2.2.7, we have

$$\frac{\partial I_s(T_1,T_2)}{\partial T_1}$$

$$= -\frac{1}{T_1^{\,2}}\ \beta_2(T_1,T_2) + \frac{1}{T_1}\ \frac{\beta(f(\beta)-1)}{\beta f(\beta)-T_1}\Bigg|_{\beta=\beta_2(T_1,T_2)}$$

$$= \left[\frac{(T_1-\beta)f(\beta)}{T_1^{\,2}(f(\beta)-1)}\ \ \frac{\partial \beta_2(T_1,T_2)}{\partial T_1}\right]_{\beta=\beta_2(T_1,T_2)}. \tag{9.3.2}$$

Now $0 < \beta_2(T_1,T_2) < T_1$ by Lemma 8.2.2.8, $f(\beta_2(T_1,T_2)) > 2$ by Lemma 8.2.2.1, and $\frac{\partial \beta_2(T_1,T_2)}{\partial T_1} > 0$ by Lemma 8.2.2.7. Hence

$$\frac{\partial I_s(T_1,T_2)}{\partial T_1} > 0.$$

We conclude that the steady-state proportion I_s of infected intermediate hosts is decreased by a decrease of either T_1 or T_2.

In order to study control of the proportion of infected intermediate hosts we deduce from (9.3.2)

$$T_1 \frac{\partial I_s(T_1,T_2)}{\partial T_1}$$

$$= \left[\frac{(T_1-\beta)f(\beta)}{T_1(f(\beta)-1)} \quad \frac{\partial \beta_2(T_1,T_2)}{\partial T_1}\right] \Bigg|_{\beta=\beta_2(T_1,T_2)} . \tag{9.3.3}$$

Furthermore we find, using Lemmas 8.2.2.6 and 8.2.2.7, that

$$T_2 \frac{\partial I_s(T_1,T_2)}{\partial T_2} = \left[\frac{T_1-\beta}{T_1} \frac{\partial \beta_2(T_1,T_2)}{\partial T_1}\right] \Bigg|_{\beta=\beta_2(T_1,T_2)} . \tag{9.3.4}$$

A comparison now shows the right-hand side of (9.3.3) to be larger than the right-hand side of (9.3.4). We conclude therefore that

$$T_1 \frac{\partial I_s(T_1,T_2)}{\partial T_1} > T_2 \frac{\partial I_s(T_1,T_2)}{\partial T_2} .$$

Using by now familiar arguments we conclude that control of the steady-state proportion of infected intermediate hosts through decrease of T_1 is more efficient than control through decrease of T_2.

9.4 The Public Health Factor

In this section we introduce and study a public health factor as a measure of steady-state public health conditions in the community. We assume that $T_2 > T_{20}$ (T_1) and that the initial data (given by f_k, m_k, and y_o, $k = 1,\ldots,N_1$) are such that the solution of Problem (8.2) approaches w_2 as $t \to \infty$.

The clinical symptoms of a definitive host harboring schistosomes are due mainly to the eggs laid by paired mature female parasites. The probability that the number of paired mature female parasites increases by 1 in a short time-interval, divided by the length of the time-interval, is taken as a measure of public health conditions in the community. We denote the time-interval by $(t,t+\Delta t)$ and use arguments in Chapter 3 to find that the above-described probability is

$$[\tfrac{1}{2}\,\nu_1\, Y(t)\,\Delta t + o(\Delta t)]\, P\{\xi_{MS}(t) > 0\}$$

$$+ [\tfrac{1}{2}\,\nu_1\, Y(t)\,\Delta t + o(\Delta t)]\, P\{\xi_{FS}(t) > 0\}.$$

Now, $Y(t) \to \dfrac{N_2}{T_1}\,\beta_2(T_1,T_2)$ as $t \to \infty$, and by (3.5.5) and (3.4.3)

$$P\{\xi_{MS}(t) > 0\} \to \tfrac{1}{2}\,\psi'(\beta_2(T_1,T_2))$$

as $t \to \infty$. The probability $P\{\xi_{FS}(t) > 0\}$ has the same limiting value.

As a measure of steady-state public health conditions in the community we now introduce a function H_s as follows:

$$H_s(\nu_1 N_2, \mu_1, T_2) = \frac{1}{2}\,\mu_1 \beta \psi'(\beta)\Big|_{\beta=\beta_2(T_1,T_2)}, \tag{9.4.1}$$

where the function $\beta_2 = \bar{\beta}_2$ is defined in Lemmas 8.2.2.6 and 8.2.2.7, and $T_1 = \dfrac{\nu_1 N_2}{\mu_1}$. We refer to the function H_s as the public health factor.

In preparation for a study of control of the public health factor, we establish the following lemma. For brevity, we omit the arguments in the function H_s.

Lemma 9.4.1

$$\nu_1 N_2 \frac{\partial H_s}{\partial(\nu_1 N_2)} > -\mu_1 \frac{\partial H_s}{\partial \mu_1} > T_2 \frac{\partial H_s}{\partial T_2} > 0.$$

Proof:

A direct differentation of (9.4.1) shows

$$\frac{\partial H_s}{\partial(\nu_1 N_2)} = \frac{1}{2} \frac{\partial \beta_2(T_1,T_2)}{\partial T_1} (\psi'(\beta)+\beta\psi''(\beta)) \Big|_{\beta=\beta_2(T_1,T_2)},$$

$$\frac{\partial H_s}{\partial \mu_1} = \frac{1}{2} \left[\beta\psi'(\beta) - T_1 \frac{\partial \beta_2(T_1,T_2)}{\partial T_1} (\psi'(\beta)+\beta\psi''(\beta))\right]\Big|_{\beta=\beta_2(T_1,T_2)},$$

and

$$\frac{\partial H_s}{\partial T_2} = \frac{1}{2}\,\mu_1\,\frac{\partial \beta_2(T_1,T_2)}{\partial T_2}\,(\psi'(\beta)+\beta\psi''(\beta))\,\Big|_{\beta=\beta_2(T_1,T_2)}.$$

The last expression here is positive, since, by Lemma 8.2.2.6,

$$\frac{\partial \beta_2(T_1,T_2)}{\partial T_2} > 0,$$

and, by Lemma 3.4.1(ii) and (iii),

$$\psi'(\beta) > 0,\ \psi''(\beta) > 0 \qquad \text{for } \beta > 0.$$

Next,

$$\nu_1 N_2\,\frac{\partial H_s}{\partial(\nu_1 N_2)} + \mu_1\,\frac{\partial H_s}{\partial \mu_1}$$

$$= \frac{1}{2}\,\mu_1\,\beta_2(T_1,T_2)\,\psi'(\beta_2(T_1,T_2)) > 0.$$

Finally, using Lemmas 8.2.2.6 and 8.2.2.7,

$$-\mu_1\,\frac{\partial H_s}{\partial \mu_1} - T_2\,\frac{\partial H_s}{\partial T_2}$$

$$= \frac{1}{2}\,\mu_1\,\frac{\partial \beta_2(T_1,T_2)}{\partial T_1}\,\left[\frac{T_1-\beta}{f(\beta)-1}\,\psi'(\beta) + \beta^2\psi''(\beta)\right]\Bigg|_{\beta=\beta_2(T_1,T_2)}$$

It is readily verified that this expression is positive. QED

From the above lemma we conclude that the public health factor is decreased by a decrease in $\nu_1 N_2$, an increase in μ_1, or a decrease in T_2. In terms of control of the public health factor we find that a decrease in $\nu_1 N_2$ is more efficient than an increase in μ_1, and that an increase in μ_1 is more efficient than a decrease in T_2.

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Mathematical models are established for two classes of parasitic diseases. In each case the disease is caused by a parasite which reaches sexual maturity inside its definitive host and which requires one intermediate host for the completion of its life cycle. In one case the mature parasite is hermaphroditic, while in the other case it has a monogamous heterosexual mating pattern.

From each model we derive an autonomous non-linear system of differential equations, whose solutions describe the course of the disease in a community in which it is endemic. We establish existence in the large, uniqueness, and boundedness of the solutions of these systems, and we study the critical points and their stability properties. From these results we deduce the influence of various biological and environmental parameters on the steady-state prevalence of the disease, on the proportion of infected intermediate hosts, and on a measure of public health. This makes it possible to compare the effectiveness of various procedures aimed at control or eradication of the disease and reveals the import of the parasite's mode of mating for the efficacy of such procedures.

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